Chapter 12

Electrodynamics and Relativity

Problem 12.1

Let $\mathbf{u}$ be the velocity of a particle in $\mathcal{S}$, $\bar{\mathbf{u}}$ its velocity in $\bar{\mathcal{S}}$, and $\mathbf{v}$ the velocity of $\bar{\mathcal{S}}$ with respect to $\mathcal{S}$. Galileo's velocity addition rule says that $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{v}$. For a free particle, $\mathbf{u}$ is constant (that's Newton's first law in $\mathcal{S}$).

(a) If $\mathbf{v}$ is constant, then $\bar{\mathbf{u}} = \bar{\mathbf{u}} - \mathbf{v}$ is also constant, so Newton's first law holds in $\bar{\mathcal{S}}$, and hence $\mathcal{S}$ is inertial.

(b) If $\bar{\mathcal{S}}$ is inertial, then $\bar{\mathbf{u}}$ is also constant, so $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}$ is constant.

Problem 12.2

(a) $m_A\mathbf{u}_A + m_B\mathbf{u}_B = m_C\mathbf{u}_C + m_D\mathbf{u}_D; \quad \mathbf{u}_i = \bar{\mathbf{u}}_i + \mathbf{v}.$

$m_A(\bar{\mathbf{u}}_A + \mathbf{v}) + m_B(\bar{\mathbf{u}}_B + \mathbf{v}) = m_C(\bar{\mathbf{u}}_C + \mathbf{v}) + m_D(\bar{\mathbf{u}}_D + \mathbf{v}),$

$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B + (m_A + m_B)\mathbf{v} = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D + (m_C + m_D)\mathbf{v}.$

Assuming *mass* is conserved, $(m_A + m_B) = (m_C + m_D)$, it follows that

$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D$, so momentum is conserved in $\bar{\mathcal{S}}$.

(b) $\frac{1}{2}m_Au_A^2 + \frac{1}{2}m_Bu_B^2 = \frac{1}{2}m_Cu_C^2 + \frac{1}{2}m_Du_D^2 \;\Rightarrow$

$$\tfrac{1}{2}m_A(\bar{u}_A^2 + 2\bar{\mathbf{u}}_A\cdot\mathbf{v} + v^2) + \tfrac{1}{2}m_B(\bar{u}_B^2 + 2\bar{\mathbf{u}}_B\cdot\mathbf{v} + v^2) = \tfrac{1}{2}m_C(\bar{u}_C^2 + 2\bar{\mathbf{u}}_C\cdot\mathbf{v} + v^2) + \tfrac{1}{2}m_D(\bar{\mathbf{u}}_D^2 + 2\bar{\mathbf{u}}_D\cdot\mathbf{v} + v^2)$$

$$\begin{aligned}&\tfrac{1}{2}m_A\bar{u}_A^2 + \tfrac{1}{2}m_B\bar{u}_B^2 + 2\mathbf{v}\cdot(m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B) + \tfrac{1}{2}v^2(m_A + m_B)\\ &\quad = \tfrac{1}{2}m_C\bar{u}_C^2 + \tfrac{1}{2}m_D\bar{u}_D^2 + 2\mathbf{v}\cdot(m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D) + \tfrac{1}{2}v^2(m_C + m_D).\end{aligned}$$

But the middle terms are equal by conservation of momentum, and the last terms are equal by conservation of mass, so $\frac{1}{2}m_A\bar{u}_A^2 + \frac{1}{2}m_B\bar{u}_B^2 = \frac{1}{2}m_C\bar{u}_C^2 + \frac{1}{2}m_D\bar{u}_D^2$. qed

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Problem 12.3

(a) $v_G = v_{AB} + v_{BC}$; $v_E = \frac{v_{AB}+v_{BC}}{1+v_{AB}v_{BC}/c^2} \approx v_G\left(1 - \frac{v_{AB}v_{BC}}{c^2}\right) \Rightarrow \frac{v_G - v_E}{v_G} = \frac{v_{AB}v_{BC}}{c^2}$.

In mi/h, $c = (186{,}000\,\text{mi/s}) \times (3600\,\text{sec/hr}) = 6.7 \times 10^8\,\text{mi/hr}$.

$\therefore\ \frac{v_G - v_E}{v_G} = \frac{(5)(60)}{(6.7\times10^8)^2} = 6.7 \times 10^{-16} \Rightarrow \boxed{6.7 \times 10^{-14}\%\text{ error,}}$ (pretty small!)

(b) $\left(\frac{1}{2}c + \frac{3}{4}c\right) / \left(1 + \frac{1}{2}\cdot\frac{3}{4}\right) = \left(\frac{5}{4}c\right) / \left(\frac{11}{8}\right) = \boxed{\frac{10}{11}c}$ (still *less* than c).

(c) To simplify the notation, let $\beta \equiv v_{AC}/c$, $\beta_1 \equiv v_{AB}/c$, $\beta_2 \equiv v_{BC}/c$. Then Eq. 12.3 says: $\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$, or:

$$\beta^2 = \frac{\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2}{(1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2)} = \frac{1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2}{(1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2)} - \frac{(1 + \beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2)}{(1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2)} = 1 - \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2} = 1 - \Delta,$$

where $\Delta \equiv (1-\beta_1^2)(1-\beta_2^2)/(1+\beta_1\beta_2)^2$ is clearly a *positive* number. So $\beta_2 < 1$, and hence $|v_{AC}| < c$. qed

Problem 12.4

(a) Velocity of bullet relative to ground: $\frac{1}{2}c + \frac{1}{3}c = \frac{5}{6}c = \frac{10}{12}c$.
Velocity of getaway car: $\frac{3}{4}c = \frac{9}{12}c$. Since $v_b > v_g$, $\boxed{\text{bullet } does \text{ reach target.}}$

(b) Velocity of bullet relative to ground: $\frac{\frac{1}{2}c+\frac{1}{3}c}{1+\frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}c}{\frac{7}{6}} = \frac{5}{7}c = \frac{20}{28}c$.
Velocity of getaway car: $\frac{3}{4}c = \frac{21}{28}c$. Since $v_g > v_b$, $\boxed{\text{bullet does } not \text{ reach target.}}$

Problem 12.5

(a) Light from the 90th clock took $\frac{90\times10^9 \text{ m}}{3\times10^8 \text{ m/s}} = 300$ s $= 5$ min to reach me, so the time I *see* on the clock is $\boxed{\text{11:55 am.}}$

(b) I *observe* $\boxed{\text{12 noon.}}$

Problem 12.6

$$\begin{cases} \text{light signal leaves } a \text{ at time } t_a'; \text{ arrives at earth at time } t_a = t_a' + d_a/c, \\ \text{light signal leaves } b \text{ at time } t_b'; \text{ arrives at earth at time } t_b = t_b' + d_b/c. \end{cases}$$

$$\therefore \Delta t = t_b - t_a = t_b' - t_a' + \frac{(d_b - d_a)}{c} = \Delta t' + \frac{(-v\Delta t'\cos\theta)}{c} = \Delta t'\left[1 - \frac{v}{c}\cos\theta\right].$$

(Here d_a is the distance from a to earth, and d_b is the distance from b to earth.)

$$\Delta s = v\Delta t'\sin\theta = \frac{v\sin\theta\,\Delta t}{(1 - v/c\cos\theta)};\quad \boxed{u = \frac{v\sin\theta}{(1-\frac{v}{c}\cos\theta)}} \text{ is the the apparent velocity.}$$

$$\frac{du}{d\theta} = \frac{v\left[(1-\frac{v}{c}\cos\theta)(\cos\theta) - \sin\theta(\frac{v}{c}\sin\theta)\right]}{(1-\frac{v}{c}\cos\theta)^2} = 0 \Rightarrow (1 - \frac{v}{c}\cos\theta)\cos\theta = \frac{v}{c}\sin^2\theta$$
$$\Rightarrow \cos\theta = \frac{v}{c}(\sin^2\theta + \cos^2\theta) = \frac{v}{c}$$

$\boxed{\theta_{\max} = \cos^{-1}(v/c).}$ At this maximal angle, $u = \frac{v\sqrt{1-v^2/c^2}}{1-v^2/c^2} = \frac{v}{\sqrt{1-v^2/c^2}}$.
As $v \to c$, $\boxed{u \to \infty,}$ because the denominator $\to 0$, even though $v < c$.

Problem 12.7

The student has not taken into account time dilation of the muon's "internal clock". In the *laboratory*, the muon lasts $\gamma\tau = \frac{\tau}{\sqrt{1-v^2/c^2}}$, where τ is the "proper" lifetime, 2×10^{-6} s. Thus

$$v = \frac{d}{\tau/\sqrt{1-v^2/c^2}} = \frac{d}{\tau}\sqrt{1-v^2/c^2}, \text{ where } d = 800 \text{ m}.$$

$$\left(\frac{\tau}{d}\right)^2 v^2 = 1 - \frac{v^2}{c^2};\quad v^2\left[\left(\frac{\tau}{d}\right)^2 + \frac{1}{c^2}\right] = 1;\quad v^2 = \frac{1}{(\tau/d)^2 + (1/c)^2}.$$

$$\frac{v^2}{c^2} = \frac{1}{1+(\tau c/d)^2};\quad \frac{\tau c}{d} = \frac{(2\times10^{-6})(3\times10^8)}{800} = \frac{6}{8} = \frac{3}{4};\quad \frac{v^2}{c^2} = \frac{1}{1+9/16} = \frac{16}{25};\quad \boxed{v = \frac{4}{5}c.}$$

Problem 12.8

(a) Rocket clock runs slow; so earth clock reads $\gamma t = \frac{1}{\sqrt{1-v^2/c^2}} \cdot 1$ hr. Here $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-9/25}} = \frac{5}{4}$.

$\therefore$ According to earth clocks signal was sent $\boxed{\text{1 hr and 15 min}}$ after take-off.

(b) By earth observer, rocket is now a distance $\left(\frac{3}{5}c\right)\left(\frac{5}{4}\right)(1\text{ hr}) = \frac{3}{4}c$ hr (three-quarters of a light hour) away. Light signal will therefore take $\frac{3}{4}$ hr to return to earth. Since it *left* 1 hr and 15 min after departure, light signal reaches earth $\boxed{\text{2 hrs after takeoff.}}$

(c) Earth clocks run slow: $t_{\text{rocket}} = \gamma \cdot (2\text{ hrs}) = \frac{5}{4} \cdot (2\text{ hrs}) = \boxed{2.5\text{ hrs.}}$

Problem 12.9

$$L_c = 2L_v;\ \frac{L_c}{\gamma_c} = \frac{L_v}{\gamma_v};\ \text{so } \frac{2}{\gamma_c} = \frac{1}{\gamma_v} = \sqrt{1-\left(\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{3}{4}};\ \frac{1}{\gamma_c^2} = 1 - \frac{v^2}{c^2} = \frac{3}{16};\ \frac{v^2}{c^2} = 1 - \frac{3}{16} = \frac{13}{16};\ \boxed{v = \frac{\sqrt{13}}{4}c.}$$

Problem 12.10

Say length of mast (at rest) is l. To an observer on the boat, height of mast is $l\sin\theta$, horizontal projection is $l\cos\theta$. To observer on dock, the former is unaffected, but the latter is Lorentz contracted to $\frac{1}{\gamma}l\cos\theta$. Therefore:

$$\tan\bar{\theta} = \frac{l\sin\theta}{\frac{1}{\gamma}l\cos\theta} = \gamma\tan\theta, \text{ or } \boxed{\tan\bar{\theta} = \frac{\tan\theta}{\sqrt{1-v^2/c^2}}.}$$

Problem 12.11

Naively, circumference/diameter $= \frac{1}{\gamma}(2\pi R)/(2R) = \pi/\gamma = \pi\sqrt{1-(\omega R/c)^2}$ — but this is nonsense. Point is: an accelerating object cannot remain rigid, in relativity. To decide what *actually* happens here, you need a specific model for the internal forces holding the disk together.

Problem 12.12

(iv) $\Rightarrow t = \frac{\bar{t}}{\gamma} + \frac{vx}{c^2}$. Put this into (i), and solve for x:

$$\bar{x} = \gamma x - \gamma v\left(\frac{\bar{t}}{\gamma} + \frac{vx}{c^2}\right) = \gamma x\left(1 - \frac{v^2}{c^2}\right) - v\bar{t} = \gamma x\frac{1}{\gamma^2} - v\bar{t} = \frac{x}{\gamma} - v\bar{t};\ \boxed{x = \gamma(\bar{x} + v\bar{t}).}\ \checkmark$$

Similarly, (i) $\Rightarrow x = \frac{\bar{x}}{\gamma} + vt$. Put this into (iv) and solve for t:

$$\bar{t} = \gamma t - \frac{\gamma v}{c^2}\left(\frac{\bar{x}}{\gamma} + vt\right) = \gamma t\left(1 - \frac{v^2}{c^2}\right) - \frac{v}{c^2}\bar{x} = \frac{t}{\gamma} - \frac{v}{c^2}\bar{x};\ \boxed{t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right).}\ \checkmark$$

Problem 12.13

Let brother's accident occur at origin, time zero, in both frames. In system $\mathcal{S}$ (Sophie's), the coordinates of Sophie's cry are $x = 5 \times 10^5$ m, $t = 0$. In system $\bar{\mathcal{S}}$ (scientist's), $\bar{t} = \gamma(t - \frac{v}{c^2}x) = -\gamma vx/c^2$. Since this is *negative*, $\boxed{\text{Sophie's cry occurred \textit{before} the accident,}}$ in $\bar{\mathcal{S}}$. $\gamma = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{13}{\sqrt{169-144}} = \frac{13}{5}$. So $\bar{t} = -\left(\frac{13}{5}\right)\left(\frac{12}{13}c\right)(5 \times 10^5)/c^2 = -12 \times 10^5/3 \times 10^8 = -4 \times 10^{-3}$. $\boxed{4 \times 10^{-3}\text{ s earlier.}}$

Problem 12.14

(a) In $\mathcal{S}$ it moves a distance dy in time dt. In $\bar{\mathcal{S}}$, meanwhile, it moves a distance $d\bar{y} = dy$ in time $d\bar{t} = \gamma(dt - \frac{v}{c^2}dx)$.

$$\therefore \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{(dy/dt)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)};\ \text{or } \boxed{\bar{u}_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)};\ \bar{u}_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}.}$$

(b) $\tan\bar{\theta} = -\dfrac{\bar{u}_y}{\bar{u}_x} = -\dfrac{u_y/\left[\gamma\left(1-\frac{vu_x}{c^2}\right)\right]}{(u_x-v)/\left(1-\frac{vu_x}{c^2}\right)} = \dfrac{1}{\gamma}\dfrac{(-u_y)}{(u_x-v)}.$

In this case $u_x = -c\cos\theta$; $u_y = c\sin\theta \Rightarrow \tan\bar{\theta} = \frac{1}{\gamma}\left(\frac{-c\sin\theta}{-c\cos\theta - v}\right)$.

$\boxed{\tan\bar{\theta} = \dfrac{1}{\gamma}\left(\dfrac{\sin\theta}{\cos\theta + v/c}\right).}$ [Compare $\tan\bar{\theta} = \gamma\frac{\sin\theta}{\cos\theta}$ in Prob. 12.10. The point is that *velocities* are sensitive not only to the transformation of *distances*, but also of *times*. That's why there is no *universal* rule for translating angles—you have to know whether it's an angle made by a *velocity* vector or a *position* vector.]

Problem 12.15

Bullet relative to ground: $\dfrac{5}{7}c$, Outlaws relative to police: $\dfrac{\frac{3}{4}c - \frac{1}{2}c}{1 - \frac{3}{4}\cdot\frac{1}{2}} = \dfrac{(1/4)c}{(5/8)} = \dfrac{2}{5}c.$

Bullet relative to outlaws: $\dfrac{\frac{5}{7}c - \frac{3}{4}c}{1 - \frac{5}{7}\cdot\frac{3}{4}} = \dfrac{-(1/28)c}{(13/28)} = -\dfrac{1}{13}c$. [Velocity of A relative to B is minus the velocity of B relative to A, so all entries below the diagonal are trivial. Note that in every case $v_{\text{bullet}} < v_{\text{outlaws}}$, so no matter how you look at it, the bad guys get away.]

speed of → relative to ↓	Ground	Police	Outlaws	Bullet	Do they escape?
Ground	0	$\frac{1}{2}c$	$\frac{3}{4}c$	$\frac{5}{7}c$	Yes
Police	$-\frac{1}{2}c$	0	$\frac{2}{5}c$	$\frac{1}{3}c$	Yes
Outlaws	$-\frac{3}{4}c$	$-\frac{2}{5}c$	0	$-\frac{1}{13}c$	Yes
Bullet	$-\frac{5}{7}c$	$-\frac{1}{3}c$	$\frac{1}{13}c$	0	Yes

Problem 12.16

(a) Moving clock runs slow, by a factor $\gamma = \frac{1}{\sqrt{1-(4/5)^2}} = \frac{5}{3}$. Since 18 years elapsed on the moving clock, $\frac{5}{3}\times 18 = 30$ years elapsed on the stationary clock. $\boxed{\text{51 years old.}}$

(b) By earth clock, it took 15 years to get there, at $\frac{4}{5}c$, so $d = \frac{4}{5}c \times 15$ years $= \boxed{12c\text{ years}}$ (12 light years).

(c) $\boxed{t = 15\text{ years}, x = 12c\text{ years.}}$

(d) $\boxed{\bar{t} = 9\text{ years}, \bar{x} = 0.}$ [She got *on* at the origin in $\bar{S}$, and rode along with $\bar{S}$, so she's *still* at the origin. If you doubt these values, use the Lorentz transformations, with x and t from (c).]

(e) Lorentz transformations: $\left\{\begin{array}{l} \tilde{x} = \gamma(x + vt) \\ \tilde{t} = \gamma(t + \frac{v}{c^2}x) \end{array}\right\}$ (note that v is *negative*, since $\tilde{S}$ is going to the *left*).

$\therefore \tilde{x} = \frac{5}{3}(12c\text{ yrs} + \frac{4}{5}c\cdot 15\text{ yrs}) = \frac{5}{3}\cdot 24c\text{ yrs} = \boxed{40c\text{ years.}}$

$\tilde{t} = \frac{5}{3}(15\text{ yrs} + \frac{4}{5}\frac{c}{c^2}\cdot 12c\text{ yrs}) = \frac{5}{3}\left(15 + \frac{48}{5}\right)\text{ yrs} = (25+16)\text{ yrs} = \boxed{\text{41 years.}}$

(f) Set her clock $\boxed{\text{ahead 32 years,}}$ from 9 to 41 ($\bar{t} \to \tilde{t}$). Return trip takes 9 years (moving time), so her clock will now read $\boxed{50}$ years at her arrival. Note that this is $\frac{5}{3}\cdot 30$ years—precisely what she would calculate if the *stay-at-home* had been the traveler, for 30 years of his own time.

(g) (i) $\bar{t} = 9$ yrs, $x = 0$. What is t? $t = \frac{v}{c^2}x + \frac{\bar{t}}{\gamma} = \frac{3}{5}\cdot 9 = \frac{27}{5} = 5.4$ years, and he started at age 21, so he's $\boxed{\text{26.4 years old.}}$ (*Younger* than the traveler (!) because to the traveler it's the stay-at-home who's moving.)

(ii) $\tilde{t} = 41$ yrs, $x = 0$. What is t? $t = \frac{\tilde{t}}{\gamma} = \frac{3}{5}\cdot 41 = \frac{123}{5} = 24.6$ years, and he started at 21, so he's $\boxed{\text{45.6 years old.}}$

(h) It will take another $\boxed{\text{5.4 years}}$ of earth time for the return, so when she gets back, she will say her twin's age is 45.6 + 5.4 = $\boxed{51}$ years—which is what we found in (a). But note that to make it work from traveler's point of view you *must* take into account the jump in *perceived* age of stay-at-home when she changes coordinates from $\bar{S}$ to $\tilde{S}$.

Problem 12.17

$$\begin{aligned}
-\bar{a}^0\bar{b}^0+\bar{a}^1\bar{b}^1+\bar{a}^2\bar{b}^2+\bar{a}^3\bar{b}^3 &= -\gamma^2(a^0-\beta a^1)(b^0-\beta b^1)+\gamma^2(a^1-\beta a^0)(b^1-\beta a^0)+a^2b^2+a^3b^3\\
&= -\gamma^2(a^0b^0-\beta\cancel{a^0}b^1-\beta\cancel{a^1}b^0+\beta^2a^1b^1-a^1b^1+\beta\cancel{a^1}b^0+\beta\cancel{a^0}b^1-\beta^2a^0b^0)+a^2b^2+a^3b^3\\
&= -\gamma^2a^0b^0(1-\beta^2)+\gamma^2a^1b^1(1-\beta^2)+a^2b^2+a^3b^3\\
&= -a^0b^0+a^1b^1+a^2b^2+a^3b^3. \quad \text{qed} \quad [\textit{Note: } \gamma^2(1-\beta^2)=1.]
\end{aligned}$$

Problem 12.18

(a) $\boxed{\begin{pmatrix} c\bar{t}\\ \bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-\beta&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}}$ (using the notation of Eq. 12.24, for best comparison).

(b) $\boxed{\Lambda=\begin{pmatrix}\gamma&0&-\gamma\beta&0\\0&1&0&0\\-\gamma\beta&0&\gamma&0\\0&0&0&1\end{pmatrix}}.$

(c) Multiply the matrices: $\Lambda=\begin{pmatrix}\bar{\gamma}&0&-\bar{\gamma}\bar{\beta}&0\\0&1&0&0\\-\bar{\gamma}\bar{\beta}&0&\bar{\gamma}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}\gamma&-\gamma\beta&0&0\\-\gamma\beta&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\boxed{\begin{pmatrix}\gamma\bar{\gamma}&-\gamma\bar{\gamma}\beta&-\bar{\gamma}\bar{\beta}&0\\-\gamma\beta&\gamma&0&0\\-\bar{\gamma}\gamma\bar{\beta}&\gamma\bar{\gamma}\beta\bar{\beta}&\bar{\gamma}&0\\0&0&0&1\end{pmatrix}}.$

$\boxed{\text{Yes,}}$ the order *does* matter. In the other order, "bars" and "no-bars" would be switched, and this would give a *different matrix*.

Problem 12.19

(a) Since $\tanh\theta=\frac{\sinh\theta}{\cosh\theta}$, and $\cosh^2\theta-\sinh^2\theta=1$, we have:

$$\gamma=\frac{1}{\sqrt{1-v^2/c^2}}=\frac{1}{\sqrt{1-\tanh^2\theta}}=\frac{\cosh\theta}{\sqrt{\cosh^2\theta-\sinh^2\theta}}=\cosh\theta\,;\ \gamma\beta=\cosh\theta\tanh\theta=\sinh\theta.$$

$$\therefore \boxed{\Lambda=\begin{pmatrix}\cosh\theta&-\sinh\theta&0&0\\-\sinh\theta&\cosh\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}.\quad \text{Compare: } R=\begin{pmatrix}\cos\phi&\sin\phi&0\\-\sin\phi&\cos\phi&0\\0&0&1\end{pmatrix}.$$

(b) $\bar{u}=\dfrac{u-v}{1-\frac{uv}{c^2}}\Rightarrow\dfrac{\bar{u}}{c}=\dfrac{(u/c)-(v/c)}{1-\left(\frac{u}{c}\right)\left(\frac{v}{c}\right)}\Rightarrow\tanh\bar{\phi}=\dfrac{\tanh\phi-\tanh\theta}{1-\tanh\phi\tanh\theta}$, where $\tanh\phi=u/c$, $\tanh\theta=v/c$; $\tanh\bar{\phi}=\bar{u}/c$. But a "trig" formula for hyperbolic functions (CRC Handbook, 18th Ed., p. 204) says:

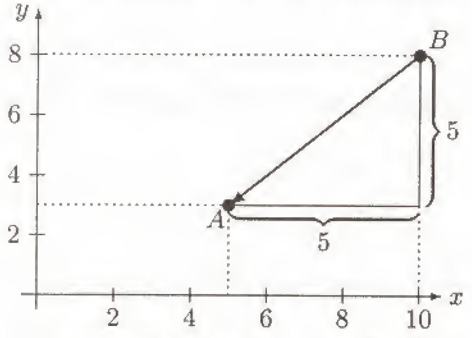
$$\frac{\tanh\phi-\tanh\theta}{1-\tanh\phi\tanh\theta}=\tanh(\phi-\theta).\quad\therefore\tanh\bar{\phi}=\tanh(\phi-\theta),\ \text{or: }\boxed{\bar{\phi}=\phi-\theta.}$$

Problem 12.20

(a) (i) $I = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -(5-15)^2 + (10-5)^2 + (8-3)^2 + (0-0)^2 = -100 + 25 + 25 = \boxed{-50.}$

(ii) $\boxed{\text{No.}}$ (In such a system $\Delta\bar{t} = 0$, so I would have to be ***positive***, which it *isn't*.)

(iii) $\boxed{\text{Yes.}}$



$\bar{S}$ travels in the direction from B toward A, making the trip in time $10/c$.

$$\therefore \mathbf{v} = \frac{-5\hat{\mathbf{x}} - 5\hat{\mathbf{y}}}{10/c} = \boxed{-\frac{c}{2}\hat{\mathbf{x}} - \frac{c}{2}\hat{\mathbf{y}}.}$$

Note that $\frac{v^2}{c^2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $v = \frac{1}{\sqrt{2}}c$, safely less than c.

(b) (i) $I = -(3-1)^2 + (5-2)^2 + 0 + 0 = -4 + 9 = \boxed{5.}$

(ii) $\boxed{\text{Yes.}}$ By Lorentz transformation: $\Delta(c\bar{t}) = \gamma\left[\Delta(ct) - \beta(\Delta x)\right]$. We want $\Delta\bar{t} = 0$, so $\Delta(ct) = \beta(\Delta x)$; or $\frac{v}{c} = \frac{\Delta(ct)}{\Delta x} = \frac{(3-1)}{(5-2)} = \frac{2}{3}$. So $\boxed{v = \frac{2}{3}c,}$ in the $+x$ direction.

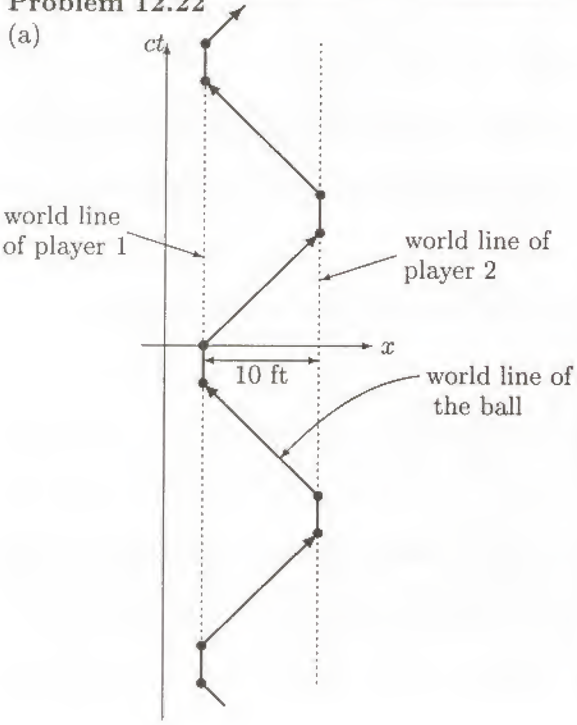
(iii) $\boxed{\text{No.}}$ (In such a system $\Delta x = \Delta y = \Delta z = 0$ so I would be negative, which it *isn't*.)

Problem 12.21

Using Eq. 12.18 (iv): $\Delta\bar{t} = \gamma(\Delta t - \frac{v}{c^2}\Delta x) = 0 \Rightarrow \Delta t = \frac{v}{c^2}\Delta x$, or $v = \frac{\Delta t}{\Delta x}c^2 = \boxed{\frac{t_B - t_A}{x_B - x_A}c^2.}$

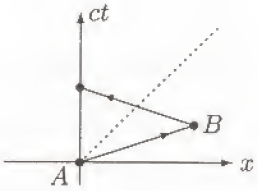
Problem 12.22

(a)



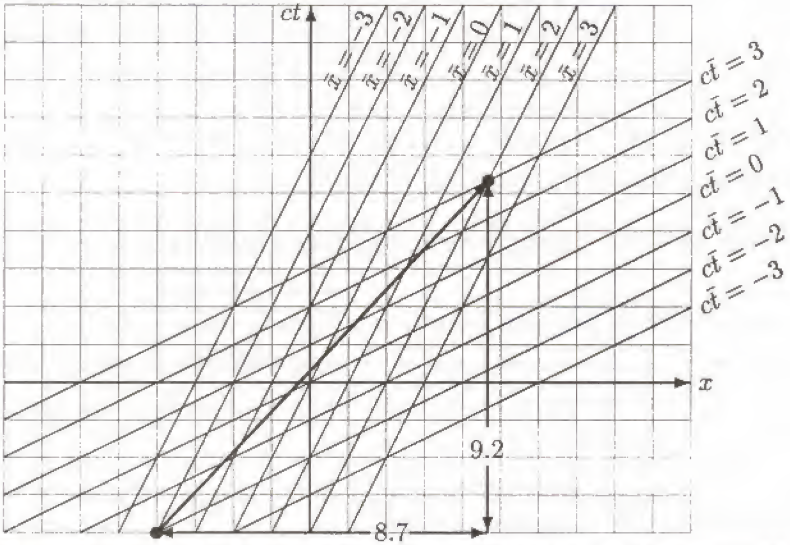
Truth is, you never *do* communicate with the other person *right now*—you communicate with the person he/she *will be* when the message gets there; and the response comes back to and older and wiser you.

(b) $\boxed{\textit{No way.}}$ It is true that a moving observer might say she arrived at B before she left A, but for the *round trip* everyone must agree that she arrives back after she set out.



Problem 12.23

(a)



(b) $\frac{c}{v}$ = slope = $\frac{9.2}{8.7}$
$\Rightarrow v = \frac{8.7}{9.2}c = \boxed{0.95c.}$

(c) $v' = \frac{4}{5}c$, so $v = \frac{\frac{4}{5}c+\frac{3}{5}c}{1+\frac{4}{5}\cdot\frac{3}{5}}$
$= \frac{(7/5)c}{(37/25)} = \boxed{\frac{35}{37}c} = 0.95c.$ ✓

Problem 12.24

(a) $(1-\frac{u^2}{c^2})\eta^2 = u^2;\ u^2(1+\frac{\eta^2}{c^2}) = \eta^2;\ \boxed{\mathbf{u} = \frac{1}{\sqrt{1+\eta^2/c^2}}\boldsymbol{\eta}.}$

(b) $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-\tanh^2\theta}} = \frac{\cosh\theta}{\sqrt{\cosh^2\theta-\sinh^2\theta}} = \cosh\theta;\quad \eta = \frac{1}{\sqrt{1-u^2/c^2}}u = \cosh\theta\, c\tanh\theta = \boxed{c\sinh\theta.}$

Problem 12.25

(a) $u_x = u_y = u\cos 45^\circ = \frac{1}{\sqrt{2}}\frac{2}{\sqrt{5}}c = \boxed{\sqrt{\frac{2}{5}}\,c.}$

(b) $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-4/5}} = \frac{\sqrt{5}}{\sqrt{5-4}} = \sqrt{5};\quad \eta = \frac{\mathbf{u}}{\sqrt{1-u^2/c^2}} \Rightarrow \boxed{\eta_x = \eta_y = \sqrt{2}\,c.}$

(c) $\eta^0 = \gamma c = \boxed{\sqrt{5}\,c.}$

(d) Eq. 12.45⇒ $\left\{\begin{array}{l} \bar{u}_x = \frac{u_x - v}{1-\frac{u_x v}{c^2}} = \frac{\sqrt{2/5}\,c-\sqrt{2/5}\,c}{1-\frac{2}{5}} = \boxed{0.} \\ \bar{u}_y = \frac{1}{\gamma}\left(\frac{u_y}{1-\frac{u_x v}{c^2}}\right) = \sqrt{1-\frac{2}{5}}\,\frac{\sqrt{2/5}\,c}{1-\frac{2}{5}} = \frac{\sqrt{2/5}}{\sqrt{3/5}}c = \boxed{\sqrt{\frac{2}{3}}\,c.} \end{array}\right.$

(e) $\bar{\eta}_x = \gamma(\eta_x - \beta\eta^0) = \sqrt{1-\frac{2}{5}}\left(\sqrt{2}\,c - \sqrt{\frac{2}{5}}\sqrt{5}\,c\right) = \boxed{0.}\quad \boxed{\bar{\eta}_y = \eta_y = \sqrt{2}\,c.}$

(f) $\frac{1}{\sqrt{1-\bar{u}^2/c^2}} = \frac{1}{\sqrt{1-(2/3)}} = \sqrt{3};\quad \bar{\eta} = \sqrt{3}\,\bar{\mathbf{u}} \Rightarrow \left\{\begin{array}{l} \bar{\eta}_x = \sqrt{3}\,\bar{u}_x = 0. \checkmark \\ \bar{\eta}_y = \sqrt{3}\,\bar{u}_y = \sqrt{2}\,c. \checkmark \end{array}\right\}$

Problem 12.26

$$\eta^\mu\eta_\mu = -(\eta^0)^2 + \eta^2 = \frac{1}{(1-u^2/c^2)}(-c^2+u^2) = -c^2\frac{(1-u^2/c^2)}{(1-u^2/c^2)} = \boxed{-c^2.}$$

Problem 12.27

(a) From Prob. 11.31 we have $\gamma = \frac{1}{b}\sqrt{b^2 + c^2t^2}$. $\therefore \tau = \int \frac{1}{\gamma} dt = b\int \frac{dt}{\sqrt{b^2+c^2t^2}} = \frac{b}{c}\ln(ct + \sqrt{b^2+c^2t^2}) + k$; at $t = 0$ we want $\tau = 0$: $0 = \frac{b}{c}\ln b + k$, so $k = -\frac{b}{c}\ln b$; $\boxed{\tau = \dfrac{b}{c}\ln\left[\dfrac{1}{b}(ct + \sqrt{b^2 + c^2t^2})\right].}$

(b) $\sqrt{x^2 - b^2} + x = be^{c\tau/b}$; $\sqrt{x^2 - b^2} = be^{c\tau/b} - x$; $x^2 - b^2 = b^2e^{2c\tau/b} - 2xbe^{c\tau/b} + x^2$; $2xbe^{c\tau/b} = b^2(1 + e^{2c\tau/b})$; $x = b(\frac{e^{c\tau/b}+e^{-c\tau/b}}{2}) = \boxed{b\cosh(c\tau/b).}$ Also from Prob. 11.31: $v = c^2t/\sqrt{b^2 + c^2t^2}$.

$v = \frac{c}{x}\sqrt{x^2 - b^2} = \frac{c}{b\cosh(c\tau/b)}\sqrt{b^2\cosh^2(c\tau/b) - b^2} = c\frac{\sqrt{\cosh^2(c\tau/b)-1}}{\cosh(c\tau/b)} = c\frac{\sinh(c\tau/b)}{\cosh(c\tau/b)} = \boxed{c\tanh\left(\dfrac{c\tau}{b}\right).}$

(c) $\eta^\mu = \gamma(c, v, 0, 0)$; $\gamma = \frac{x}{b} = \cosh\frac{c\tau}{b}$, so $\eta^\mu = \cosh\frac{c\tau}{b}\left(c, c\tanh\frac{c\tau}{b}, 0, 0\right) = \boxed{c\left(\cosh\dfrac{c\tau}{b}, \sinh\dfrac{c\tau}{b}, 0, 0\right).}$

Problem 12.28

(a) $m_Au_A + m_Bu_B = m_Cu_C + m_Du_D$; $\quad u_i = \dfrac{\bar{u}_i + v}{1 + (\bar{u}_iv/c^2)}$.

$$m_A\frac{\bar{u}_A + v}{1 + (\bar{u}_Av/c^2)} + m_B\frac{\bar{u}_B + v}{1 + (\bar{u}_Bv/c^2)} = m_C\frac{\bar{u}_C + v}{1 + (\bar{u}_Cv/c^2)} + m_D\frac{\bar{u}_D + v}{1 + (\bar{u}_Dv/c^2)}.$$

This time, because the denominators are all different, we *cannot* conclude that

$m_A\bar{u}_A + m_B\bar{u}_B = m_C\bar{u}_C + m_D\bar{u}_D$.

As an explicit counterexample, suppose all the masses are equal, and $u_A = -u_B = v$; $u_C = u_D = 0$. This is a symmetric "completely inelastic" collision in $\mathcal{S}$, and momentum is clearly conserved ($0 = 0$). But the Einstein velocity addition rule gives $\bar{u}_A = 0$, $\bar{u}_B = -2u/(1 + u^2/c^2)$, $\bar{u}_C = \bar{u}_D = -u$, so in $\bar{\mathcal{S}}$ the (incorrectly defined) momentum is *not* conserved:

$$m\left(\frac{-2u}{1 + u^2/c^2}\right) \neq -2mu.$$

(b) $m_A\eta_A + m_B\eta_B = m_C\eta_C + m_D\eta_D$; $\quad \eta_i = \gamma(\bar{\eta}_i + \beta\bar{\eta}_i^0)$. (The inverse Lorentz transformation.)

$m_A\gamma(\bar{\eta}_A + \beta\bar{\eta}_A^0) + m_B\gamma(\bar{\eta}_B + \beta\bar{\eta}_B^0) = m_C\gamma(\bar{\eta}_C + \beta\bar{\eta}_C^0) + m_D\gamma(\bar{\eta}_D + \beta\bar{\eta}_D^0)$. The gamma's cancel:

$m_A\bar{\eta}_A + m_B\bar{\eta}_B + \beta(m_A\bar{\eta}_A^0 + m_B\bar{\eta}_B^0) = m_C\bar{\eta}_C + m_D\bar{\eta}_D + \beta(m_C\bar{\eta}_C^0 + m_D\bar{\eta}_D^0)$.

But $m_i\eta_i^0 = p_i^0 = E_i/c$, so if $\boxed{\textit{energy}\text{ is conserved}}$ in $\bar{\mathcal{S}}$ ($\bar{E}_A + \bar{E}_B = \bar{E}_C + \bar{E}_D$), then so too is the momentum (correctly defined):

$m_A\bar{\eta}_A + m_B\bar{\eta}_B = m_C\bar{\eta}_C + m_D\bar{\eta}_D$. qed

Problem 12.29

$\gamma mc^2 - mc^2 = nmc^2 \Rightarrow \gamma = n + 1 = \frac{1}{\sqrt{1-u^2/c^2}} \Rightarrow 1 - \frac{u^2}{c^2} = \frac{1}{(n+1)^2}$.

$\therefore \frac{u^2}{c^2} = 1 - \frac{1}{(n+1)^2} = \frac{n^2+2n+1-1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$; $\boxed{u = \dfrac{\sqrt{n(n+2)}}{n+1}c.}$

Problem 12.30

$E_T = E_1 + E_2 + \cdots$; $\quad p_T = p_1 + p_2 + \cdots$; $\quad \bar{p}_T = \gamma(p_T - \beta E_T/c) = 0 \Rightarrow \beta = v/c = p_Tc/E_T$.

$v = c^2p_T/E_T = \boxed{c^2(p_1 + p_2 + \cdots)/(E_1 + E_2 + \cdots).}$

Problem 12.31

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi}c^2 = \gamma m_\mu c^2 \Rightarrow \gamma = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi m_\mu} = \frac{1}{\sqrt{1 - v^2/c^2}};\ 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2};$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{4m_\pi^2m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{m_\pi^4 + 2m_\pi^2m_\mu^2 + m_\mu^4 - 4m_\pi^2m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{(m_\pi^2 - m_\mu^2)^2}{(m_\pi^2 + m_\mu^2)^2};\ v = \boxed{\left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)c.}$$

Problem 12.32

Initial momentum: $E^2 - p^2c^2 = m^2c^4 \Rightarrow p^2c^2 = (2mc^2)^2 - m^2c^4 = 3m^2c^4 \Rightarrow p = \sqrt{3}\,mc$.
Initial energy: $2mc^2 + mc^2 = 3mc^2$.

Each is conserved, so final energy is $3mc^2$, final momentum is $\sqrt{3}\,mc$.

$$E^2 - p^2c^2 = (3mc^2)^2 - (\sqrt{3}\,mc)^2c^2 = 6m^2c^4 = M^2c^4 \Rightarrow \boxed{M = \sqrt{6}\,m} \approx 2.5m.$$

(In this process some kinetic energy was converted into rest energy, so $M > 2m$.)

$$v = \frac{pc^2}{E} = \frac{\sqrt{3}\,mc\,c^2}{3mc^2} = \boxed{\frac{c}{\sqrt{3}}.}$$

Problem 12.33

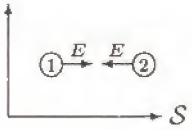
First calculate pion's energy: $E^2 = p^2c^2 + m^2c^4 = \frac{9}{16}m^2c^4 + m^2c^4 = \frac{25}{16}m^2c^4 \Rightarrow E = \frac{5}{4}mc^2$.

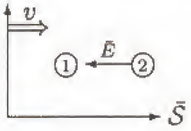
$$\left.\begin{aligned} &\text{Conservation of energy:} && \tfrac{5}{4}mc^2 = E_A + E_B \\ &\text{Conservation of momentum:} && \tfrac{3}{4}mc^2 = p_A + p_B = \tfrac{E_A}{c} - \tfrac{E_B}{c} \Rightarrow \tfrac{3}{4}mc^2 = E_A - E_B \end{aligned}\right\} 2E_A = 2mc^2.$$

$$\Rightarrow \boxed{E_A = mc^2;} \quad \boxed{E_B = \frac{1}{4}mc^2.}$$

Problem 12.34

Classically, $E = \frac{1}{2}mv^2$. In a colliding beam experiment, the relative velocity (classically) is *twice* the velocity of either one, so the relative energy is $4E$.





Let $\bar{\mathcal{S}}$ be the system in which ① is at rest. Its speed v, relative to $\mathcal{S}$, is just the speed of ① in $\mathcal{S}$.

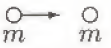
$\bar{p}^0 = \gamma(p^0 - \beta p^1) \Rightarrow \frac{\bar{E}}{c} = \gamma\left(\frac{E}{c} - \beta p\right)$, where p is the momentum of ② in $\mathcal{S}$.
$E = \gamma Mc^2$, so $\gamma = \frac{E}{Mc^2}$; $p = -\gamma Mv = -\gamma M\beta c$; $\bar{E} = \gamma\left(\frac{E}{c} + \beta\gamma M\beta c\right)c = \gamma(E + \gamma Mc^2\beta^2)$.
$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow 1 - \beta^2 = \frac{1}{\gamma^2} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2} = \frac{\gamma^2-1}{\gamma^2}$; $\bar{E} = \frac{E}{Mc^2}E + \left[\left(\frac{E}{Mc^2}\right)^2 - 1\right]Mc^2$.

$$\bar{E} = \frac{E^2}{Mc^2} + \frac{E^2}{Mc^2} - Mc^2; \quad \boxed{\bar{E} = \frac{2E^2}{Mc^2} - Mc^2.}$$

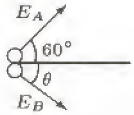
For $E = 30$ GeV and $Mc^2 = 1$ GeV, we have $\bar{E} = \frac{(2)(900)}{1} - 1 = 1800 - 1 = \boxed{1799\text{ GeV}} = \boxed{60E.}$

Problem 12.35

One photon is impossible, because in the "center of momentum" frame (Prob. 12.30) we'd be left with a photon *at rest*, whereas photons *have* to travel at speed c.



(before)



(after)

$$\left\{\begin{aligned} &\text{Cons. of energy:} \ \sqrt{p_0c^2 + m^2c^4} + mc^2 = E_A + E_B. \\ &\text{Cons. of mom.:} \left\{\begin{aligned} &\text{horizontal:} && p_0 = \tfrac{E_A}{c}\cos 60^\circ + \tfrac{E_B}{c}\cos\theta \Rightarrow E_B\cos\theta = p_0c - \tfrac{1}{2}E_A, \\ &\text{vertical:} && 0 = \tfrac{E_A}{c}\sin 60^\circ - \tfrac{E_B}{c}\sin\theta \Rightarrow E_B\sin\theta = \tfrac{\sqrt{3}}{2}E_A; \end{aligned}\right\} \text{square and add:} \end{aligned}\right.$$

$$E_B^2(\cos^2\theta + \sin^2\theta) = p_0c^2 - p_0cE_A + \frac{1}{4}E_A^2 + \frac{3}{4}E_A^2$$

$$\Rightarrow E_B^2 = p_0c^2 - p_0cE_A + E_A^2 = \left[\sqrt{p_0^2c^2 + m^2c^4} + mc^2 - E_A\right]^2$$

$$= p_0c^2 + m^2c^4 + 2\sqrt{p_0^2c^2 + m^2c^4}(mc^2 - E_A) + m^2c^4 - 2E_Amc^2 + E_A^2. \text{ Or:}$$

$$-p_0cE_A = 2m^2c^4 + 2mc^2\sqrt{p_0^2c^2 + m^2c^4} - 2E_A\sqrt{p_0^2c^2 + m^2c^4} - 2E_Amc^2;$$

$$E_A(mc^2 + \sqrt{p_0^2c^2 + m^2c^4} - p_0c/2) = m^2c^4 + mc^2\sqrt{p_0c^2 + m^2c^4};$$

$$E_A = mc^2\frac{(mc^2 + \sqrt{p_0^2c^2 + m^2c^4})}{(mc^2 + \sqrt{p_0^2c^2 + m^2c^4} - p_0c/2)} \cdot \frac{(mc^2 - \sqrt{p_0^2c^2 + m^2c^4} - p_0c/2)}{(mc^2 - \sqrt{p_0^2c^2 + m^2c^4} - p_0c/2)}$$

$$= mc^2\frac{(\cancel{m^2}c^4 - p_0^2c^2 - \cancel{m^2}c^4 - \frac{1}{2}p_0mc^3 - \frac{p_0c}{2}\sqrt{p_0^2c^2 + m^2c^4})}{(\cancel{m^2}c^4 - p_0mc^3 + \frac{p_0^2c^2}{4} - p_0^2c^2 - \cancel{m^2}c^4)} = \boxed{\frac{mc^2}{2}\frac{(mc + 2p_0 + \sqrt{p_0^2 + m^2c^2})}{(mc + \frac{3}{4}p_0)}.}$$

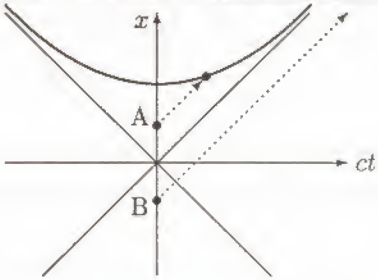
Problem 12.36

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} = m\left\{\frac{\frac{d\mathbf{u}}{dt}}{\sqrt{1 - u^2/c^2}} + \mathbf{u}\left(-\frac{1}{2}\right)\frac{-\frac{1}{c^2}2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}}{(1 - u^2/c^2)^{3/2}}\right\}$$

$$= \frac{m}{\sqrt{1 - u^2/c^2}}\left\{\mathbf{a} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2 - u^2)}\right\}. \quad \text{qed}$$

Problem 12.37

At constant force you go in "hyperbolic" motion. Photon A, which left the origin at $t < 0$, catches up with you, but photon B, which passes the origin at $t > 0$, never does.



Problem 12.38

(a)

$$\alpha^0 = \frac{d\eta_0}{d\tau} = \frac{d\eta_0}{dt}\frac{dt}{d\tau} = \left[\frac{d}{dt}\left(\frac{c}{\sqrt{1 - u^2/c^2}}\right)\right]\frac{1}{\sqrt{1 - u^2/c^2}}$$

$$= \frac{c}{\sqrt{1 - u^2/c^2}}\left(-\frac{1}{2}\right)\frac{(-\frac{1}{c^2})2\mathbf{u}\cdot\mathbf{a}}{(1 - u^2/c^2)^{3/2}} = \boxed{\frac{1}{c}\frac{\mathbf{u}\cdot\mathbf{a}}{(1 - u^2/c^2)^2}.}$$

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\eta}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{\eta}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}}\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = \frac{1}{\sqrt{1 - u^2/c^2}}\left\{\frac{\mathbf{a}}{\sqrt{1 - u^2/c^2}} + \mathbf{u}(-\frac{1}{2})\frac{-\frac{1}{c^2}2\mathbf{u}\cdot\mathbf{a}}{(1 - u^2/c^2)^{3/2}}\right\}$$

$$= \boxed{\frac{1}{(1 - u^2/c^2)}\left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2 - u^2)}\right].}$$

(b) $$\alpha_\mu\alpha^\mu = -(\alpha^0)^2 + \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = -\frac{1}{c^2}\frac{(\mathbf{u}\cdot\mathbf{a})^2}{(1-u^2/c^2)^4} + \frac{1}{(1-u^2/c^2)^4}\left[\mathbf{a}\left(1-\frac{u^2}{c^2}\right)+\frac{1}{c^2}\mathbf{u}(\mathbf{u}\cdot\mathbf{a})\right]^2$$

$$= \frac{1}{(1-u^2/c^2)^4}\left\{-\frac{1}{c^2}(\mathbf{u}\cdot\mathbf{a})^2 + a^2\left(1-\frac{u^2}{c^2}\right)^2 + \frac{2}{c^2}\left(1-\frac{u^2}{c^2}\right)(\mathbf{u}\cdot\mathbf{a})^2 + \frac{1}{c^4}u^2(\mathbf{u}\cdot\mathbf{a})^2\right\}$$

$$= \frac{1}{(1-u^2/c^2)^4}\left\{a^2\left(1-\frac{u^2}{c^2}\right)^2 + \frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2}\underbrace{\left(-1+2-2\frac{u^2}{c^2}+\frac{u^2}{c^2}\right)}_{(1-\frac{u^2}{c^2})}\right\}$$

$$= \boxed{\frac{1}{(1-u^2/c^2)^2}\left[a^2+\frac{(\mathbf{u}\cdot\mathbf{a})^2}{(c^2-u^2)}\right].}$$

(c) $\eta^\mu\eta_\mu = -c^2$, so $\frac{d}{d\tau}(\eta^\mu\eta_\mu) = \alpha^\mu\eta_\mu + \eta^\mu\alpha_\mu = 2\alpha^\mu\eta_\mu = 0$, so $\boxed{\alpha^\mu\eta_\mu = 0.}$

(d) $K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}(m\eta^\mu) = \boxed{m\alpha^\mu.}$ $\boxed{K^\mu\eta_\mu = m\alpha^\mu\eta_\mu = 0.}$

Problem 12.39

$K_\mu K^\mu = -(K^0)^2 + \mathbf{K}\cdot\mathbf{K}$. From Eq. 12.70, $\mathbf{K}\cdot\mathbf{K} = \frac{F^2}{(1-u^2/c^2)}$. From Eq. 12.71:

$$K^0 = \frac{1}{c}\frac{dE}{d\tau} = \frac{1}{c\sqrt{1-u^2/c^2}}\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{mc}{\sqrt{1-u^2/c^2}}\left[-\frac{1}{2}\frac{(-1/c^2)}{(1-u^2/c^2)^{3/2}}2\mathbf{u}\cdot\mathbf{a}\right] = \frac{m}{c}\frac{(\mathbf{u}\cdot\mathbf{a})}{(1-u^2/c^2)^2}.$$

But (Eq. 12.73): $\mathbf{u}\cdot\mathbf{F} = uF\cos\theta = \frac{m}{\sqrt{1-u^2/c^2}}\left[(\mathbf{u}\cdot\mathbf{a})+\frac{u^2(\mathbf{u}\cdot\mathbf{a})}{c^2(1-u^2/c^2)}\right] = \frac{m(\mathbf{u}\cdot\mathbf{a})}{(1-u^2/c^2)^{3/2}}$, so

$$K^0 = \frac{uF\cos\theta}{c\sqrt{1-u^2/c^2}};\quad K_\mu K^\mu = \frac{F^2}{(1-u^2/c^2)} - \frac{u^2F^2\cos^2\theta}{c^2(1-u^2/c^2)} = \left[\frac{1-(u^2/c^2)\cos^2\theta}{(1-u^2/c^2)}\right]F^2.\quad \text{qed}$$

Problem 12.40

$$\mathbf{F} = \frac{m}{\sqrt{1-u^2/c^2}}\left[\mathbf{a}+\frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2-u^2}\right] = q(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \Rightarrow \mathbf{a}+\frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2-u^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}(\mathbf{E}+\mathbf{u}\times\mathbf{B}).$$

Dot in $\mathbf{u}$: $(\mathbf{u}\cdot\mathbf{a}) + \frac{u^2(\mathbf{u}\cdot\mathbf{a})}{c^2(1-u^2/c^2)} = \frac{\mathbf{u}\cdot\mathbf{a}}{(1-u^2/c^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}[\mathbf{u}\cdot\mathbf{E}+\underbrace{\mathbf{u}\cdot(\mathbf{u}\times\mathbf{B})}_{=0}]$;

$$\therefore \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2-u^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}\frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{E})}{c^2}.\quad \text{So } \mathbf{a} = \frac{q}{m}\sqrt{1-u^2/c^2}\left[\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{1}{c^2}\mathbf{u}(\mathbf{u}\cdot\mathbf{E})\right].\quad \text{qed}$$

Problem 12.41

One way to see it is to look back at the general formula for **E** (Eq. 10.29). For a uniform infinite plane of charge, moving at constant velocity in the plane, $\dot{\mathbf{J}} = 0$ and $\dot{\rho} = 0$, while ρ (or rather, σ) is independent of t (so retardation does nothing). Therefore the field is exactly the same as it would be for a plane at rest (except that σ itself is altered by Lorentz contraction).

A more elegant argument exploits the fact that **E** is a *vector* (whereas **B** is a *pseudo*vector). This means that any given component changes sign if the configuration is reflected in a plane perpendicular to that direction. But in Fig. 12.35(b), if we reflect in the xy plane the configuration is unaltered, so the z component of **E** would have to stay the *same*. Therefore it must in fact be zero. (By contrast, if you reflect in a plane perpendicular to the y direction the charges trade places, so it is perfectly appropriate that the y component of $\mathbf{E}$ should reverse its sign.)

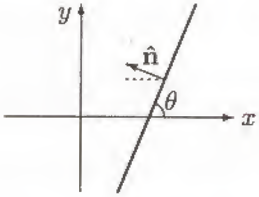
Problem 12.42

(a) Field is σ_0/ϵ_0, and it points perpendicular to the positive plate, so:

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}(\cos 45^\circ\,\hat{\mathbf{x}} + \sin 45^\circ\,\hat{\mathbf{y}}) = \boxed{\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}(-\hat{\mathbf{x}} + \hat{\mathbf{y}}).}$$

(b) From Eq. 12.108, $E_x = E_{x_0} = -\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}$; $E_y = \gamma E_{y_0} = \gamma\frac{\sigma_0}{\sqrt{2}\,E_0}$. So $\boxed{\mathbf{E} = \dfrac{\sigma_0}{\sqrt{2}\,\epsilon_0}(-\hat{\mathbf{x}} + \gamma\,\hat{\mathbf{y}}).}$

(c) From Prob. 12.10: $\tan\theta = \gamma$, so $\boxed{\theta = \tan^{-1}\gamma.}$



(d) Let $\hat{\mathbf{n}}$ be a unit vector perpendicular to the plates in $\mathcal{S}$—evidently $\hat{\mathbf{n}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$; $|E| = \frac{\sigma_0}{\sqrt{2}\,\epsilon_0}\sqrt{1+\gamma^2}$.

So the angle ϕ between $\hat{\mathbf{n}}$ and $\mathbf{E}$ is:

$$\frac{\mathbf{E}\cdot\hat{\mathbf{n}}}{|E|} = \cos\phi = \frac{1}{\sqrt{1+\gamma^2}}(\sin\theta + \gamma\cos\theta) = \frac{\cos\theta}{\sqrt{1+\gamma^2}}(\tan\theta + \gamma) = \frac{2\gamma}{\sqrt{1+\gamma^2}}\cos\theta$$

But $\gamma = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \sqrt{\frac{1}{\cos^2\theta} - 1} \Rightarrow \frac{1}{\cos}{}^2\theta = \gamma^2 + 1 \Rightarrow \cos\theta = \frac{1}{\sqrt{1+\gamma^2}}$. So $\boxed{\cos\phi = \left(\dfrac{2\gamma}{1+\gamma^2}\right).}$

Evidently the field is $\boxed{\textit{not}}$ perpendicular to the plates in $\mathcal{S}$.

Problem 12.43

(a) $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q(1-v^2/c^2)}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}}\dfrac{\hat{\mathbf{R}}}{R^2}$ (Eq. 12.92) $\Rightarrow$

$$\begin{aligned}\int \mathbf{E}\cdot d\mathbf{a} &= \frac{q(1-v^2/c^2)}{4\pi\epsilon_0}\int\frac{R^2\sin\theta\,d\theta\,d\phi}{R^2(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}}\\ &= \frac{q(1-v^2/c^2)}{4\pi\epsilon_0}2\pi\int_0^\pi\frac{\sin\theta\,d\theta}{(1-\frac{v^2}{c^2}\sin^2\theta)^{3/2}}.\quad \text{Let } u=\cos\theta, \text{ so } du = -\sin\theta\,d\theta,\ \sin^2\theta = 1-u^2.\\ &= \frac{q(1-v^2/c^2)}{2\epsilon_0}\int_{-1}^{1}\frac{du}{[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}u^2]^{3/2}} = \frac{q(1-v^2/c^2)}{2\epsilon_0}\left(\frac{c}{v}\right)^3\int_{-1}^{1}\frac{du}{\left(\frac{c^2}{v^2}-1+u^2\right)^{3/2}}.\end{aligned}$$

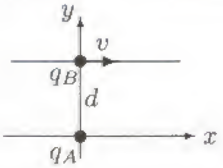
The integral is: $\left.\dfrac{u}{(\frac{c^2}{v^2}-1)\sqrt{\frac{c^2}{v^2}-1+u^2}}\right|_{-1}^{+1} = \dfrac{2}{(\frac{c^2}{v^2}-1)\frac{c}{v}} = \left(\dfrac{v}{c}\right)^3\dfrac{2}{(1-v^2/c^2)}.$

So $\displaystyle\int \mathbf{E}\cdot d\mathbf{a} = \frac{q(1-v^2/c^2)}{2\epsilon_0}\left(\frac{c}{v}\right)^3\left(\frac{v}{c}\right)^3\frac{2}{(1-v^2/c^2)} = q.$ ✓

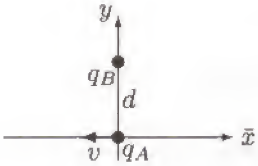
(b) Using Eq. 12.111 and Eq. 12.92, $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \dfrac{1}{\mu_0}\dfrac{1}{4\pi\epsilon_0}\dfrac{\mu_0}{4\pi}\dfrac{q^2(1-v^2/c^2)^2 v\sin\theta}{R^4(1-\frac{v^2}{c^2}\sin^2\theta)^3}\underbrace{(\hat{\mathbf{R}}\times\hat{\boldsymbol{\phi}})}_{-\hat{\boldsymbol{\theta}}}$;

$$\mathbf{S} = \boxed{-\frac{q^2}{16\pi^2\epsilon_0}\frac{(1-v^2/c^2)^2 v\sin\theta}{R^4(1-\frac{v^2}{c^2}\sin^2\theta)^3}\hat{\boldsymbol{\theta}}.}$$

Problem 12.44



(a) Fields of A at B: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$; $\mathbf{B} = 0$. So force on q_B is $\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_Aq_B}{d^2}\hat{\mathbf{y}}.}$



(b) (i) From Eq. 12.68: $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_Aq_B}{d^2}\hat{\mathbf{y}}.}$ (*Note:* here the particle is at rest in $\bar{\mathcal{S}}$.)

(ii) From Eq. 12.92, with $\theta = 90°$: $\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}}\frac{1}{d^2}\hat{\mathbf{y}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$ (this also follows from Eq. 12.108).

$\bar{\mathbf{B}} \neq 0$, but since $v_B = 0$ in $\bar{\mathcal{S}}$, there is no magnetic force anyway, and $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_Aq_B}{d^2}\hat{\mathbf{y}}}$ (as before).

Problem 12.45

Here $\theta = 90°$, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{y}}$, $\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$, $\mathscr{r} = r$, so (using $c^2 = 1/\mu_0\epsilon_0$):

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\gamma}{r^2}\hat{\mathbf{y}},\quad \mathbf{B} = -\frac{q}{4\pi\epsilon_0}\frac{v}{c^2}\frac{\gamma}{r^2}\hat{\mathbf{z}},\quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

Note that $(E^2 - B^2c^2) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2\gamma^2\left(1-\frac{v^2}{c^2}\right) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2$ is invariant, because it doesn't depend on v. We can use this as a check.

System A: $v_A = v$, so $\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\bar{\gamma}}{r^2}\hat{\mathbf{y}}$, $\mathbf{B} = -\frac{q}{4\pi\epsilon_0}\frac{v}{c^2}\frac{\bar{\gamma}}{r^2}\hat{\mathbf{z}}$, where $\bar{\gamma} = \frac{1}{\sqrt{1-v^2/c^2}}$.

$$\mathbf{F} = q[\mathbf{E} + (-v\hat{\mathbf{x}})\times\mathbf{B}] = -\frac{q^2}{4\pi\epsilon_0}\frac{\bar{\gamma}}{r^2}\left[\hat{\mathbf{y}} - \frac{v^2}{c^2}(\hat{\mathbf{x}}\times\hat{\mathbf{z}})\right] = -\frac{q^2}{4\pi\epsilon_0}\frac{\bar{\gamma}}{r^2}\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}}.$$

System B: $v_B = \frac{v+v}{1+v^2/c^2} = \frac{2v}{(1+v^2/c^2)}$

$$\gamma_B = \frac{1}{\sqrt{1-\frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{(1+v^2/c^2)}{\sqrt{1-2\frac{v^2}{c^2}+\frac{v^4}{c^4}}} = \frac{(1+v^2/c^2)}{(1-v^2/c^2)} = \bar{\gamma}^2\left(1+\frac{v^2}{c^2}\right);\ v_b\gamma_B = 2v\bar{\gamma}^2.$$

$$\therefore \mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\bar{\gamma}^2\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}};\ \mathbf{B} = -\frac{q}{4\pi\epsilon_0}\frac{2v}{c^2}\frac{\bar{\gamma}^2}{r^2}\hat{\mathbf{z}}.$$

[*Check*: $E^2 - B^2c^2 = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2\bar{\gamma}^4\left(1+\frac{2v^2}{c^2}+\frac{v^4}{c^4}-\frac{4v^2}{c^2}\right) = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2\bar{\gamma}^4\frac{1}{\bar{\gamma}^4} = \left(\frac{q}{4\pi\epsilon_0 r}\right)^2$. ✓]

$\mathbf{F} = q\mathbf{E} = -\frac{q^2}{4\pi\epsilon_0}\frac{\bar{\gamma}^2}{r^2}\left(1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}}$. ($+q$ at rest ⇒ no magnetic force). [*Check*: Eq. 12.68 ⇒ $F_A = \frac{1}{\bar{\gamma}}F_B$. ✓]

System C: $v_C = 0$. $\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{y}}$; $\mathbf{B} = 0$; $\mathbf{F} = q\mathbf{E} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{y}}$.

[The relative velocity of B and C is $2v/(1+v^2/c^2)$, and the corresponding γ is $\bar{\gamma}^2(1+v^2/c^2)$. So Eq. 12.68 ⇒ $F_C = \frac{1}{\bar{\gamma}^2(1+v^2/c^2)}F_B$. ✓]

Summary:

$$\left.\begin{array}{|c|c|c|}\hline (-\frac{q}{4\pi\epsilon_0 r^2})\gamma\,\hat{\mathbf{y}} & (-\frac{q}{4\pi\epsilon_0 r^2})\gamma^2(1+\frac{v^2}{c^2})\hat{\mathbf{y}} & (-\frac{q}{4\pi\epsilon_0 r^2})\hat{\mathbf{y}} \\ \hline (-\frac{q}{4\pi\epsilon_0 r^2})\frac{v}{c^2}\gamma\,\hat{\mathbf{z}} & (-\frac{q}{4\pi\epsilon_0 r^2})\frac{2v}{c^2}\gamma^2\hat{\mathbf{z}} & 0 \\ \hline (-\frac{q^2}{4\pi\epsilon_0 r^2})\gamma(1+\frac{v^2}{c^2})\hat{\mathbf{y}} & (-\frac{q^2}{4\pi\epsilon_0 r^2})\gamma^2(1+\frac{v^2}{c^2})\hat{\mathbf{y}} & (-\frac{q}{4\pi\epsilon_0 r^2})\hat{\mathbf{y}} \\ \hline \end{array}\right\}\gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

Problem 12.46

(a) From Eq. 12.108:

$$\begin{aligned}\bar{\mathbf{E}}\cdot\bar{\mathbf{B}} &= \bar{E}_x\bar{B}_x + \bar{E}_y\bar{B}_y + \bar{E}_z\bar{B}_z = E_xB_x + \gamma^2(E_y - vB_z)(B_y + \frac{v}{c^2}E_z) + \gamma(E_z + vB_y)(B_z - \frac{v}{c^2}E_y) \\ &= E_xB_x + \gamma^2\{E_yB_y + \frac{v}{c^2}\cancel{E_y}E_z - v\cancel{B_y}B_z - \frac{v^2}{c^2}E_zB_z + E_zB_z - \frac{v}{c^2}\cancel{E_y}E_z + v\cancel{B_y}B_z - \frac{v^2}{c^2}E_yB_y\} \\ &= E_xB_x + \gamma^2\left[E_yB_y\left(1-\frac{v^2}{c^2}\right) + E_zB_z\left(1-\frac{v^2}{c^2}\right)\right] = E_xB_x + E_yB_y + E_zB_z = \mathbf{E}\cdot\mathbf{B}.\quad\text{qed}\end{aligned}$$

(b) $\bar{E}^2 - c^2\bar{B}^2 = [E_x^2 + \gamma^2(E_y - vB_z)^2 + \gamma^2(E_z + vB_y)^2] - c^2[B_x^2 + \gamma^2(B_y + \frac{v}{c^2}E_z)^2 + \gamma^2(B_z - \frac{v}{c^2}E_y)^2]$

$$\begin{aligned}&= E_x^2 + \gamma^2(E_y^2 - 2E_y\cancel{v}B_z + v^2B_z^2 + E_z^2 + 2E_z\cancel{v}B_y + v^2B_y^2 - c^2B_y^2 - c^2 2\frac{v}{\cancel{c^2}}B_yE_z \\ &\quad - c^2\frac{v^2}{c^4}E_z^2 - c^2B_z^2 + c^2 2\frac{v}{\cancel{c^2}}B_zE_y \quad - c^2\frac{v^2}{c^4}E_y^2) - c^2B_x^2 \\ &= E_x^2 - c^2B_x^2 + \gamma^2\left[E_y^2\left(1-\frac{v^2}{c^2}\right) + E_z^2\left(1-\frac{v^2}{c^2}\right) - c^2B_y^2\left(1-\frac{v^2}{c^2}\right) - c^2B_z^2\left(1-\frac{v^2}{c^2}\right)\right] \\ &= (E_x^2 + E_y^2 + E_z^2) - c^2(B_x^2 + B_y^2 + B_z^2) = E^2 - B^2c^2.\quad\text{qed}\end{aligned}$$

(c) $\boxed{\text{No.}}$ For if $\mathbf{B} = 0$ in one system, then $(E^2 - c^2B^2)$ is *positive*. Since it is invariant, it must be positive in *any* system. Therefore $\mathbf{E} \neq 0$ in *all* systems.

Problem 12.47

(a) Making the appropriate modifications in Eq. 9.48 (and picking $\delta = 0$ for convenience),

$$\boxed{\mathbf{E}(x,y,z,t) = E_0\cos(kx-\omega t)\,\hat{\mathbf{y}},\quad \mathbf{B}(x,y,z,t) = \frac{E_0}{c}\cos(kx-\omega t)\,\hat{\mathbf{z}},\quad \text{where } k \equiv \frac{\omega}{c}.}$$

(b) Using Eq. 12.108 to transform the fields:

$$\bar{E}_x = \bar{E}_z = 0,\quad \bar{E}_y = \gamma(E_y - vB_z) = \gamma E_0\left[\cos(kx-\omega t) - \frac{v}{c}\cos(kx-\omega t)\right] = \alpha E_0\cos(kx-\omega t),$$

$$\bar{B}_x = \bar{B}_y = 0,\quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) = \gamma E_0\left[\frac{1}{c}\cos(kx-\omega t) - \frac{v}{c^2}\cos(kx-\omega t)\right] = \alpha\frac{E_0}{c}\cos(kx-\omega t),$$

where $\boxed{\alpha \equiv \gamma\left(1-\frac{v}{c}\right) = \sqrt{\frac{1-v/c}{1+v/c}}.}$

Now the inverse Lorentz transformations (Eq. 12.19) $\Rightarrow x = \gamma(\bar{x} + v\bar{t})$ and $t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$, so

$$kx - \omega t = \gamma\left[k(\bar{x} + v\bar{t}) - \omega\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)\right] = \gamma\left[\left(k - \frac{\omega v}{c^2}\right)\bar{x} - (\omega - kv)\bar{t}\right] = \bar{k}\bar{x} - \bar{\omega}\bar{t},$$

where (recalling that $k = \omega/c$): $\bar{k} \equiv \gamma\left(k - \frac{\omega v}{c^2}\right) = \gamma k(1 - v/c) = \alpha k$ and $\bar{\omega} \equiv \gamma\omega(1 - v/c) = \alpha\omega$.

Conclusion:
$$\boxed{\begin{gathered}\bar{\mathbf{E}}(\bar{x},\bar{y},\bar{z},\bar{t}) = \bar{E}_0\cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\,\hat{\mathbf{y}}, \quad \bar{\mathbf{B}}(\bar{x},\bar{y},\bar{z},\bar{t}) = \frac{\bar{E}_0}{c}\cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\,\hat{\mathbf{z}}, \\ \text{where } \bar{E}_0 = \alpha E_0, \quad \bar{k} = \alpha k, \quad \bar{\omega} = \alpha\omega, \quad \text{and } \alpha \equiv \sqrt{\frac{1 - v/c}{1 + v/c}}.\end{gathered}}$$

(c) $\boxed{\bar{\omega} = \omega\sqrt{\frac{1 - v/c}{1 + v/c}}.}$ This is the $\boxed{\text{Doppler shift}}$ for light. $\bar{\lambda} = \frac{2\pi}{\bar{k}} = \frac{2\pi}{\alpha k} = \frac{\lambda}{\alpha}$. The velocity of the wave in $\bar{S}$ is $\bar{v} = \frac{\bar{\omega}}{2\pi}\bar{\lambda} = \frac{\omega}{\lambda} = \boxed{c.}$ $\boxed{\text{Yup,}}$ this is exactly what I expected (the velocity of a light wave is the same in any inertial system).

(d) Since intensity goes like E^2, the ratio is $\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \boxed{\frac{1 - v/c}{1 + v/c}.}$

Dear Al,

The amplitude, frequency, and intensity of the light wave will all $\boxed{\text{decrease to zero}}$ as you run faster and faster. It'll get so faint you won't be able to see it, and so red-shifted even your night-vision goggles won't help. But it'll still be going 3 × 10^8 m/s relative to you. Sorry about that.

Sincerely,

David

Problem 12.48

$$\bar{t}^{02} = \Lambda^0_\lambda\Lambda^2_\sigma t^{\lambda\sigma} = \Lambda^0_0\Lambda^2_2 t^{02} + \Lambda^0_1\Lambda^2_2 t^{12} = \gamma t^{02} + (-\gamma\beta)t^{12} = \gamma(t^{02} - \beta t^{12}).$$
$$\bar{t}^{03} = \Lambda^0_\lambda\Lambda^3_\sigma t^{\lambda\sigma} = \Lambda^0_0\Lambda^3_3 t^{03} + \Lambda^0_1\Lambda^3_3 t^{13} = \gamma t^{03} + (-\gamma\beta)t^{13} = \gamma(t^{03} - \beta t^{13}) = \gamma(t^{03} + \beta t^{31}).$$
$$\bar{t}^{23} = \Lambda^2_\lambda\Lambda^3_\sigma t^{\lambda\sigma} = \Lambda^2_2\Lambda^3_3 t^{23} = t^{23}.$$
$$\bar{t}^{31} = \Lambda^3_\lambda\Lambda^1_\sigma t^{\lambda\sigma} = \Lambda^3_3\Lambda^1_0 t^{30} + \Lambda^3_3\Lambda^1_1 t^{31} = (-\gamma\beta)t^{30} + \gamma t^{31} = \gamma(t^{31} + \beta t^{03}).$$
$$\bar{t}^{12} = \Lambda^1_\lambda\Lambda^2_\sigma t^{\lambda\sigma} = \Lambda^1_0\Lambda^2_2 t^{02} + \Lambda^1_1\Lambda^2_2 t^{12} = (-\gamma\beta)t^{02} + \gamma t^{12} = \gamma(t^{12} - \beta t^{02}).$$

Problem 12.49

Suppose $t^{\nu\mu} = \pm t^{\mu\nu}$ (+ for symmetric, − for antisymmetric).

$$\bar{t}^{\kappa\lambda} = \Lambda^\kappa_\mu\Lambda^\lambda_\nu t^{\mu\nu}$$

$\bar{t}^{\lambda\kappa} = \Lambda^\lambda_\mu\Lambda^\kappa_\nu t^{\mu\nu} = \Lambda^\lambda_\nu\Lambda^\kappa_\mu t^{\nu\mu}$ [Because μ and ν are both summed from $0 \to 3$, it doesn't matter which we call μ and and which call ν.]

$= \Lambda^\kappa_\mu\Lambda^\lambda_\mu(\pm t^{\mu\nu})$ [I used the symmetry of $t^{\mu\nu}$, and wrote the Λ's in the other order.]

$= \pm\bar{t}^{\kappa\lambda}$. qed

Problem 12.50

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} - F^{10}F^{10} - F^{20}F^{20} - F^{30}F^{30} \\ &\quad + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \\ &= -(E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 - (E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 + B_z^2 + B_y^2 + B_z^2 + B_x^2 + B_y^2 + B_x^2 \\ &= 2B^2 - 2E^2/c^2 = \boxed{2\Big(B^2 - \frac{E^2}{c^2}\Big),} \end{aligned}$$

which, apart from the constant factor $-\frac{2}{c^2}$, is the invariant we found in Prob. 12.46(b).

$\boxed{G^{\mu\nu}G_{\mu\nu} = 2(E^2/c^2 - B^2)}$ (the same invariant).

$$\begin{aligned} F^{\mu\nu}G_{\mu\nu} &= -2\left(F^{01}G^{01} + F^{02}G^{02} + F^{03}G^{03}\right) + 2\left(F^{12}G^{12} + F^{13}G^{13} + F^{23}G^{23}\right) \\ &= -2\left(\frac{1}{c}E_xB_x + \frac{1}{c}E_yB_y + \frac{1}{c}E_zB_z\right) 2\left[B_z(-E_z/c) + (-B_y)(E_y/c) + B_x(-E_x/c)\right] \\ &= -\frac{2}{c}(\mathbf{E}\cdot\mathbf{B}) - \frac{2}{c}(\mathbf{E}\cdot\mathbf{B}) = \boxed{-\frac{4}{c}(\mathbf{E}\cdot\mathbf{B}),} \end{aligned}$$

which, apart from the factor $-4/c$, is the invariant of Prob. 12.46(a). [These are, incidentally, the *only* fundamental invariants you can construct from **E** and **B**.]

Problem 12.51

$$\left.\begin{aligned} \mathbf{E} &= \tfrac{1}{4\pi\epsilon_0}\tfrac{2\lambda}{x}\hat{\mathbf{x}} = \tfrac{\mu_0}{2\pi}\tfrac{\lambda c^2}{x}\hat{\mathbf{x}} \\ \mathbf{B} &= \tfrac{\mu_0}{4\pi}\tfrac{2\lambda v}{x}\hat{\mathbf{y}} = \tfrac{\mu_0}{2\pi}\tfrac{\lambda v}{x}\hat{\mathbf{y}} \end{aligned}\right\} \boxed{F^{\mu\nu} = \frac{\mu_0\lambda}{2\pi x}\begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & -v \\ 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 \end{pmatrix}}.$$

Problem 12.52

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$. Differentiate: $\partial_\mu\partial_\nu F^{\mu\nu} = \mu_0\partial_\mu J^\mu$.

But $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ (the combination is *symmetric*) while $F^{\nu\mu} = -F^{\mu\nu}$ (*anti*symmetric).

$\therefore \partial_\mu\partial_\nu F^{\mu\nu} = 0$. [Why? Well, these indices are both summed from $0 \to 3$, so it doesn't matter which we call μ, which ν: $\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\nu\partial_\mu F^{\nu\mu} = \partial_\mu\partial_\nu(-F^{\mu\nu}) = -\partial_\mu\partial_\nu F^{\mu\nu}$. But if a quantity is equal to minus itself, it must be zero.] *Conclusion:* $\partial_\mu J^\mu = 0$. qed

Problem 12.53

We know that $\partial_\nu G^{\mu\nu} = 0$ is equivalent to the two homogeneous Maxwell equations, $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$. All we have to show, then, is that $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ is *also* equivalent to them. Now this equation stands for 64 separate equations ($\mu = 0 \to 3$, $\nu = 0 \to 3$, $\lambda = 0 \to 3$, and $4 \times 4 \times 4 = 64$). But many of them are redundant, or trivial.

Suppose two indices are the same (say, $\mu = \nu$). Then $\partial_\lambda F_{\mu\mu} + \partial_\mu F_{\mu\lambda} + \partial_\mu F_{\lambda\mu} = 0$. But $F_{\mu\mu} = 0$ and $F_{\mu\lambda} = -F_{\lambda\mu}$, so this is trivial: $0 = 0$. To get anything significant, then, μ, ν, λ must all be *different*. They could be *all spatial* ($\mu, \nu, \lambda = 1, 2, 3 = x, y, z$ — or some permutation thereof), or *one temporal* and *two spatial* ($\mu = 0$, $\nu, \lambda = 1, 2$ or $2, 3$, or $1, 3$ — or some permutation). Let's examine these two cases separately.

All spatial: say, $\mu = 1$, $\nu = 2$, $\lambda = 3$ (other permutations yield the same equation, or minus it).

$$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = 0 \Rightarrow \frac{\partial}{\partial z}(B_z) + \frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) = 0 \Rightarrow \nabla\cdot\mathbf{B} = 0.$$

One temporal: say, $\mu = 0$, $\nu = 1$, $\lambda = 2$ (other permutations of these indices yield the same result, or minus it).

$$\partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0 \Rightarrow \frac{\partial}{\partial y}\left(-\frac{E_x}{c}\right) + \frac{\partial}{\partial(ct)}(B_z) + \frac{\partial}{\partial x}\left(\frac{E_y}{c}\right) = 0,$$

or $-\frac{\partial B_z}{\partial t} + (\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}) = 0$, which is the z component of $-\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\mathbf{E}$. (If $\mu = 0$, $\nu = 1$, $\lambda = 2$, we get the y component; for $\nu = 2$, $\lambda = 3$ we get the x component.)

Conclusion: $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ is equivalent to $\nabla\cdot\mathbf{B} = 0$ and $\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$, and hence to $\partial_\nu G^{\mu\nu} = 0$. qed

Problem 12.54

$K^0 = q\eta_\nu F^{0\nu} = q(\eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}) = q(\boldsymbol{\eta}\cdot\mathbf{E})/c = \boxed{\frac{q}{c}\gamma\mathbf{u}\cdot\mathbf{E}.}$ Now from Eq. 12.71 we know that $K^0 = \frac{1}{c}\frac{dW}{d\tau}$, where W is the energy of the particle. Since $d\tau = \frac{1}{\gamma}dt$, we have:

$$\frac{1}{c}\gamma\frac{dW}{dt} = \frac{q}{c}\gamma(\mathbf{u}\cdot\mathbf{E}) \Rightarrow \boxed{\frac{dW}{dt} = q(\mathbf{u}\cdot\mathbf{E}).}$$

This says *the power delivered to the particle is force* ($q\mathbf{E}$) *times velocity* ($\mathbf{u}$) — which is as it *should* be.

Problem 12.55

$$\overline{\partial^0\phi} = \frac{\partial}{\partial\bar{x}_0}\phi = -\frac{1}{c}\frac{\partial}{\partial\bar{t}}\phi = -\frac{1}{c}(\frac{\partial\phi}{\partial t}\frac{\partial t}{\partial\bar{t}} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\bar{t}} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\bar{t}} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\bar{t}}).$$

From Eq. 12.19, we have: $\frac{\partial t}{\partial\bar{t}} = \gamma$, $\frac{\partial x}{\partial\bar{t}} = \gamma v$, $\frac{\partial y}{\partial\bar{t}} = \frac{\partial z}{\partial\bar{t}} = 0$.

So $\overline{\partial^0\phi} = -\frac{1}{c}\gamma(\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial x})$ or (since $ct = x^0 = -x_0$): $\overline{\partial^0\phi} = \gamma(\frac{\partial\phi}{\partial x_0} - \frac{v}{c}\frac{\partial\phi}{\partial x^1}) = \gamma\left[(\partial^0\phi) - \beta(\partial^1\phi)\right]$.

$$\overline{\partial^1\phi} = \frac{\partial}{\partial\bar{x}}\phi = \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial x^1} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\bar{x}} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\bar{x}} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\bar{x}} = \gamma\frac{v}{c^2}\frac{\partial\phi}{\partial t} + \gamma\frac{\partial\phi}{\partial x} = \gamma(\frac{\partial\phi}{\partial x_1} - \frac{v}{c}\frac{\partial\phi}{\partial x_0}) = \gamma\left[(\partial^1\phi) - \beta(\partial^0\phi)\right].$$

$$\overline{\partial^2\phi} = \frac{\partial\phi}{\partial\bar{y}} = \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial\bar{y}} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\bar{y}} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\bar{y}} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\bar{y}} = \frac{\partial\phi}{\partial y} = \partial^2\phi.$$

$$\overline{\partial^3\phi} = \frac{\partial\phi}{\partial\bar{z}} = \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial\bar{z}} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\bar{z}} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\bar{z}} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\bar{z}} = \frac{\partial\phi}{\partial z} = \partial^3\phi.$$

Conclusion: $\partial^\mu\phi$ transforms in the same way as a^μ (Eq. 12.27)—and hence is a contravariant 4-vector. qed

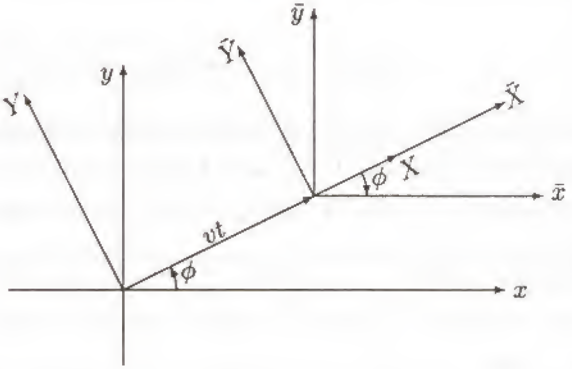
Problem 12.56

According to Prob. 12.53, $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ is equivalent to Eq. 12.129. Using Eq. 12.132, we find (in the notation of Prob. 12.55):

$$\begin{aligned}\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} &= \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \\ &= \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) \\ &= (\partial_\lambda\partial_\mu A_\nu - \partial_\mu\partial_\lambda A_\nu) + (\partial_\mu\partial_\nu A_\lambda - \partial_\nu\partial_\mu A_\lambda) + (\partial_\nu\partial_\lambda A_\mu - \partial_\lambda\partial_\nu A_\mu) = 0. \quad \text{qed}\end{aligned}$$

[Note that $\partial_\lambda\partial_\mu A_\nu = \frac{\partial^2 A_\nu}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2 A_\nu}{\partial x^\mu \partial x^\lambda} = \partial_\mu\partial_\lambda A_\nu$, by equality of cross-derivatives.]

Problem 12.57



Step 1: rotate from xy to XY, using Eq. 1.29:

$$X = \cos\phi\, x + \sin\phi\, y$$
$$Y = -\sin\phi\, x + \cos\phi\, y$$

Step 2: Lorentz-transform from XY to $\bar{X}\bar{Y}$, using Eq. 12.18:

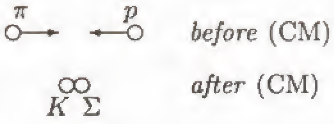
$$\bar{X} = \gamma(X - vt) = \gamma[\cos\phi\, x + \sin\phi\, y - \beta ct]$$
$$\bar{Y} = Y = -\sin\phi\, x + \cos\phi\, y$$
$$\bar{Z} = Z = z$$
$$c\bar{t} = \gamma(ct - \beta X) = \gamma\big[ct - \beta(\cos\phi\, x + \sin\phi\, y)\big]$$

Step 3: Rotate from $\bar{X}\bar{Y}$ to $\bar{x}\bar{y}$, using Eq. 1.29 with negative ϕ:

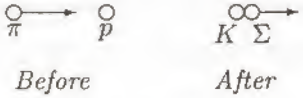
$$\begin{aligned}\bar{x} &= \cos\phi\,\bar{X} - \sin\phi\,\bar{Y} = \gamma\cos\phi[\cos\phi\, x + \sin\phi\, y - \beta ct] - \sin\phi[-\sin\phi\, x + \cos\phi\, y]\\ &= (\gamma\cos^2\phi + \sin^2\phi)x + (\gamma - 1)\sin\phi\cos\phi\, y - \gamma\beta\cos\phi\,(ct)\\ \bar{y} &= \sin\phi\,\bar{X} + \cos\phi\,\bar{Y} = \gamma\sin\phi(\cos\phi\, x + \sin\phi\, y - \beta ct) + \cos\phi(-\sin\phi\, c + \cos\phi\, y)\\ &= (\gamma - 1)\sin\phi\cos\phi\, x + (\gamma\sin^2\phi + \cos^2\phi)y - \gamma\beta\sin\phi\,(ct)\end{aligned}$$

In matrix form: $$\begin{pmatrix} c\bar{t}\\ \bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \boxed{\begin{pmatrix} \gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi & 0\\ -\gamma\beta\cos\phi & (\gamma\cos^2\phi + \sin^2\phi) & (\gamma - 1)\sin\phi\cos\phi & 0\\ -\gamma\beta\sin\phi & (\gamma - 1)\sin\phi\cos\phi & (\gamma\sin^2\phi + \cos^2\phi) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}} \begin{pmatrix} ct\\ x\\ y\\ z\end{pmatrix}.$$

Problem 12.58



In center-of-momentum system, threshold occurs when incident energy is *just* sufficient to cover the *rest* energy of the resulting particles, with none "wasted" as kinetic energy. Thus, in lab system, we want the outgoing K and Σ to have the *same velocity*, at threshold:



Initial momentum: p_π; initial energy of π: $E^2 - p^2c^2 = m^2c^4 \Rightarrow E_\pi^2 = m_\pi^2c^4 + p_\pi^2c^2$.

Total initial energy: $m_pc^2 + \sqrt{m_\pi^2c^4 + p_\pi^2c^2}$. These are also the final energy and momentum: $E^2 - p^2c^2 = (m_K + m_\Sigma)^2c^4$.

$$\left(m_pc^2 + \sqrt{m_\pi^2c^4 + p_\pi^2c^2}\right)^2 - p_\pi^2c^2 = (m_K + m_\Sigma)^2c^4$$

$$m_p^2\cancel{c^4} + \frac{2m_pc^2}{c^4}\sqrt{m_\pi^2c^2 + p_\pi^2}\,c + m_\pi^2\cancel{c^4} + \cancel{p_\pi^2c^2} - \cancel{p_\pi^2c^2} = (m_K + m_\Sigma)^2\cancel{c^4}$$

$$\frac{2m_p}{c}\sqrt{m_\pi^2c^2 + p_\pi^2} = (m_K + m_\Sigma)^2 - m_p^2 - m_\pi^2$$

$$(m_\pi^2c^2+p_\pi^2)\frac{4m_p^2}{c^2} = (m_K+m_\Sigma)^4 - 2(m_p^2+m_\pi^2)(m_K+m_\Sigma)^2 + m_p^4 + m_\pi^4 + 2m_p^2m_\pi^2$$

$$\frac{4m_p^2}{c^2}p_\pi^2 = (m_K+m_\Sigma)^4 - 2(m_p^2+m_\pi^2)(m_K+m_\Sigma)^2 + (m_p^2-m_\pi^2)^2$$

$$\boxed{p_\pi = \frac{c}{2m_p}\sqrt{(m_K+m_\Sigma)^4 - 2(m_p^2+m_\pi^2)(m_K+m_\Sigma)^2 + (m_p^2-m_\pi^2)^2}}$$

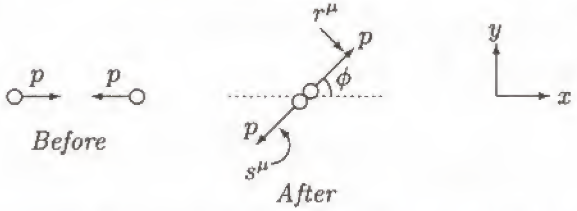
$$= \tfrac{1}{(2m_pc^2)c}\sqrt{(m_Kc^2+m_\Sigma c^2)^4 - 2\left[(m_pc^2)^2+(m_\pi c^2)^2\right](m_Kc^2+m_\Sigma c^2)^2 + \left[(m_pc^2)^2-(m_\pi c^2)^2\right]^2}$$

$$= \tfrac{1}{2c(900)}\sqrt{(1700)^4 - 2\left[(900)^2+(150)^2\right](1700)^2 + \left[(900)^2-(150)^2\right]^2}$$

$$= \tfrac{1}{1800c}\sqrt{(8.35\times10^{12}) - (4.81\times10^{12}) + (0.62\times10^{12})} = \tfrac{1}{1800c}(2.04\times10^6) = \boxed{1133\ \text{MeV}/c.}$$

Problem 12.59

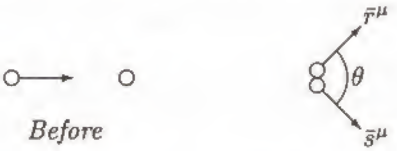
In CM:



(p = magnitude of 3-momentum in CM, ϕ = CM scattering angle)

Outgoing 4-momenta: $r^\mu = \left(\frac{E}{c}, p\cos\phi, p\sin\phi, 0\right)$; $s^\mu = \left(\frac{E}{c}, -p\cos\phi, -p\sin\phi, 0\right)$.

In Lab:



Problem: calculate θ, in terms of p, ϕ.

Lorentz transformation: $\bar{r}_x = \gamma(r_x - \beta r^0)$; $\bar{r}_y = r_y$; $\bar{s}_x = \gamma(s_x - \beta s^0)$; $\bar{s}_y = s_y$.

Now $E = \gamma mc^2$; $p = -\gamma mv$ (v here is to the *left*); $E^2 - p^2c^2 = m^2c^4$, so $\beta = -\frac{pc}{E}$.
$\therefore \bar{r}_x = \gamma\left(p\cos\phi + \frac{pc}{E}\frac{E}{c}\right) = \gamma p(1+\cos\phi)$; $\bar{r}_y = p\sin\phi$; $\bar{s}_x = \gamma p(1-\cos\phi)$; $\bar{s}_y = -p\sin\phi$.

$$\cos\theta = \frac{\bar{\mathbf{r}}\cdot\bar{\mathbf{s}}}{\bar{r}\bar{s}} = \frac{\gamma^2p^2(1-\cos^2\phi) - p^2\sin^2\phi}{\sqrt{\left[\gamma^2p^2(1+\cos\phi)^2 + p^2\sin^2\phi\right]\left[\gamma^2p^2(1-\cos\phi)^2 + p^2\sin^2\phi\right]}}$$

$$= \frac{(\gamma^2-1)\sin^2\phi}{\sqrt{\left[\gamma^2(1+\cos\phi)^2+\sin^2\phi\right]\left[\gamma^2(1-\cos\phi)^2+\sin^2\phi\right]}}$$

$$= \frac{(\gamma^2-1)}{\sqrt{\left[\gamma^2\left(\frac{1+\cos\phi}{\sin\phi}\right)^2+1\right]\left[\gamma^2\left(\frac{1-\cos\phi}{\sin\phi}\right)^2+1\right]}} = \frac{(\gamma^2-1)}{\sqrt{\left(\gamma^2\cot^2\frac{\phi}{2}+1\right)\left(\gamma^2\tan^2\frac{\phi}{2}+1\right)}}$$

$$\cos\theta = \frac{\omega}{\sqrt{\left(1+\cot^2\frac{\phi}{2}+\omega\cot^2\frac{\phi}{2}\right)\left(1+\tan^2\frac{\phi}{2}+\omega\tan^2\frac{\phi}{2}\right)}} \qquad (\text{where } \omega \equiv \gamma^2-1)$$

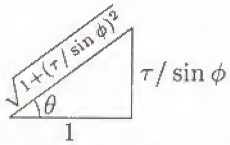
$$= \frac{\omega}{\sqrt{\left(\csc^2\frac{\phi}{2}+\omega\cot^2\frac{\phi}{2}\right)\left(\sec^2\frac{\phi}{2}+\omega\tan^2\frac{\phi}{2}\right)}} = \frac{\omega\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\sqrt{\left(1+\omega\cos^2\frac{\phi}{2}\right)\left(1+\omega\sin^2\frac{\phi}{2}\right)}}$$

$$= \frac{\frac{1}{2}\omega\sin\phi}{\sqrt{\left[1+\frac{1}{2}\omega(1+\cos\phi)\right]\left[1+\frac{1}{2}\omega(1-\cos\phi)\right]}} = \frac{\sin\phi}{\sqrt{\left[\left(\frac{2}{\omega}+1\right)+\cos\phi\right]\left[\left(\frac{2}{\omega}+1\right)-\cos\phi\right]}}$$

$$= \frac{\sin\phi}{\sqrt{\left(\frac{2}{\omega}+1\right)^2-\cos^2\phi}} = \frac{\sin\phi}{\sqrt{\frac{4}{\omega^2}+\frac{4}{\omega}+\sin^2\phi}} = \frac{1}{\sqrt{1+(\tau/\sin\phi)^2}}, \text{ where } \tau^2 = \frac{4}{\omega^2}+\frac{4}{\omega}.$$

$\sin\theta = \frac{\tau}{\sin\phi}$. $\tau^2 = \frac{4}{\omega^2}(1+\omega) = \frac{4}{(\gamma^2-1)^2}\gamma^2$, so $\tan\theta = \frac{2\gamma}{(\gamma^2-1)\sin\phi}$.

Or, since $(\gamma^2-1) = \gamma^2\left(1-\frac{1}{\gamma^2}\right) = \gamma^2\frac{v^2}{c^2}$, $\boxed{\tan\theta = \dfrac{2c^2}{\gamma v^2\sin\phi}}$.



Problem 12.60

$\frac{dp}{d\tau} = K$ (a constant) $\Rightarrow \frac{dp}{dt}\frac{dt}{d\tau} = K$. But $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}$; $p = \frac{mu}{\sqrt{1-u^2/c^2}}$.

$\therefore \frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{K}{m}\sqrt{1-u^2/c^2}$. Multiply by $\frac{dt}{dx} = \frac{1}{u}$:

$$\frac{dt}{dx}\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{d}{dx}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) = \frac{K}{m}\frac{\sqrt{1-u^2/c^2}}{u}. \text{ Let } w = \frac{u}{\sqrt{1-u^2/c^2}}.$$

$$\frac{dw}{dx} = \frac{K}{m}\frac{1}{w}\ ; \quad w\frac{dw}{dx} = \frac{1}{2}\frac{d}{dx}w^2 = \frac{k}{m}\ ; \quad \frac{d(w^2)}{dx} = \frac{2K}{m} \Rightarrow d(w^2) = \frac{2K}{m}(dx).$$

$\therefore w^2 = \frac{2K}{m}x+$ constant. But at $t=0$, $x=0$ and $u=0$ (so $w=0$), and hence the constant is 0.

$$w^2 = \frac{2K}{m}x = \frac{u^2}{1-u^2/c^2}\ ; \quad u^2 = \frac{2Kx}{m} - \frac{2Kx}{mc^2}u^2\ ; \quad u^2\left(1+\frac{2Kx}{mc^2}\right) = \frac{2Kx}{m}.$$

$$u^2 = \frac{2Kx/m}{1+\frac{2Kx}{mc^2}} = \frac{c^2}{1+\left(\frac{mc^2}{2Kx}\right)}\ ; \quad \frac{dx}{dt} = \frac{c}{\sqrt{1+\left(\frac{mc^2}{2Kx}\right)}}\ ; \quad ct = \int\sqrt{1+\left(\frac{mc^2}{2Kx}\right)}\,dx.$$

Let $\frac{mc^2}{2K} \equiv a^2$; $\quad ct = \int\frac{\sqrt{x+a^2}}{\sqrt{x}}\,dx$. $\quad$ Let $x \equiv y^2$; $dx = 2y\,dy$; $\sqrt{x} = y$.

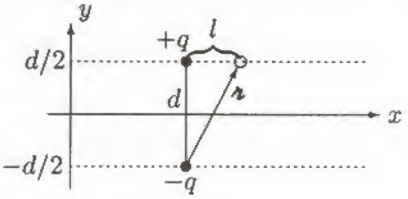
$$ct = \int\frac{\sqrt{y^2+a^2}}{y}2y\,dy = 2\int\sqrt{y^2+a^2}\,dy = \left[y\sqrt{y^2+a^2}+a^2\ln(y+\sqrt{y^2+a^2})\right] + \text{ constant.}$$

At $t=0$, $x=0 \Rightarrow y=0$, so $0 = a^2\ln a+$ constant $\Rightarrow$ constant $= -a^2\ln a$.

$$\therefore ct = y\sqrt{y^2+a^2}+a^2\ln\left(y/a+\sqrt{(y/a)^2+1}\right) = a^2\left[\left(\frac{y}{a}\right)\sqrt{\left(\frac{y}{a}\right)^2+1}+\ln\left(\frac{y}{a}+\sqrt{\left(\frac{y}{a}\right)^2+1}\right)\right].$$

Let: $z \equiv y/a = \sqrt{x}\sqrt{\frac{2K}{mc^2}} = \sqrt{\frac{2Kx}{mc^2}}$. Then $\boxed{\dfrac{2Kt}{mc} = z\sqrt{1+z^2}+\ln(z+\sqrt{1+z^2})}$.

Problem 12.61



(a) $x(t) = \frac{c}{\alpha}\left[\sqrt{1+(\alpha t)^2} - 1\right]$, where $\alpha = \frac{F}{mc}$. The force of $+q$ on $-q$ will be the mirror image of the force of $-q$ on $+q$ (in the x axis), so the *net* force is in the x direction (the net *magnetic* force is zero). All we need is the x component of $\mathbf{E}$.

The field at $+q$ due to $-q$ is: (Eq. 10.65)

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[\mathbf{u}(c^2 - v^2) + \mathbf{u}(\boldsymbol{\mathscr{r}}\cdot\mathbf{a}) - \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\right].$$

$\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v} \Rightarrow u_x = c\frac{l}{\mathscr{r}} - v = \frac{1}{\mathscr{r}}(cl - v\mathscr{r})$; $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = (c\mathscr{r} - lv)$; $\boldsymbol{\mathscr{r}}\cdot\mathbf{a} = la$. So:

$$E_x = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(c\mathscr{r} - vl)^3}\left[\frac{1}{\mathscr{r}}(cl - v\mathscr{r})(c^2 - v^2) + \underbrace{\frac{1}{\mathscr{r}}(cl - \cancel{v\mathscr{r}})la - a(c\mathscr{r} - \cancel{lv})}_{\frac{1}{\mathscr{r}}ca(l^2 - \mathscr{r}^2) = -cad^2/\mathscr{r}}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r} - vl)^3}\left[(cl - v\mathscr{r})(c^2 - v^2) - cad^2.\right]$$

The *force* on $+q$ is qE_x, and there is an equal force on $-q$, so the net force on the dipole is:

$$\boxed{\mathbf{F} = -\frac{2q^2}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r} - lv)^3}\left[(cl - v\mathscr{r})(c^2 - v^2) - cad^2\right]\hat{\mathbf{x}}.}$$

It remains to determine $\mathscr{r}$, l, v, and a, and plug these in.

$$v(t) = \frac{dx}{dt} = = \frac{c}{\alpha}\frac{1}{2}\frac{1}{\sqrt{1+(\alpha t)^2}}2\alpha^2 t = \frac{c\alpha t}{\sqrt{1-(\alpha t)^2}}\ ;\ v = v(t_r) = \frac{c\alpha t_r}{T},\ \text{where } T \equiv \sqrt{1+(\alpha t_r)^2}.$$

$$a(t_r) = \frac{dv}{dt_r} = \frac{c\alpha}{T} + c\alpha t_r\left(-\frac{1}{2}\right)\frac{2\alpha^2 t_r}{T^3} = \frac{c\alpha}{T^3}\left[1 + (\alpha t_r)^2 - (\alpha t_r)^2\right] = \frac{c\alpha}{T^3}.$$

Now calculate t_r: $c^2(t - t_r)^2 = \mathscr{r}^2 = l^2 + d^2$; $l = x(t) - x(t_r) = \frac{c}{\alpha}\left[\sqrt{1+(\alpha t)^2} - \sqrt{1+(\alpha t_r)^2}\right]$, so
$\cancel{t^2} - 2tt_r + \cancel{t_r^2} = \frac{1}{\alpha^2}\left[1 + (\cancel{\alpha t})^2 + 1 + (\alpha \cancel{t_r})^2 - 2\sqrt{1+(\alpha t)^2}\sqrt{1+(\alpha t_r)^2}\right] + (d/c)^2$
(★) $\sqrt{1+(\alpha t)^2}\sqrt{1+(\alpha t_r)^2} = 1 + \alpha^2 tt_r + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2$. Square both sides:

$$\cancel{1} + (\alpha t)^2 + (\alpha t_r)^2 + \alpha^4\cancel{t^2}t_r^2 = \cancel{1} + \alpha^4\cancel{t^2}t_r^2 + \frac{1}{4}\left(\frac{\alpha d}{c}\right)^4 + 2\alpha^2 tt_r + \left(\frac{\alpha d}{c}\right)^2 + \alpha^2 tt_r\left(\frac{\alpha d}{c}\right)^2$$

$$t^2 + t_r^2 - 2tt_r - tt_r\left(\frac{\alpha d}{c}\right)^2 - \left(\frac{d}{c}\right)^2 - \frac{\alpha^2}{4}\left(\frac{d}{c}\right)^4 = 0.$$

At this point we *could* solve for t_r in terms of t, but since v and a are already expressed in terms of t_r it is simpler to solve for t (in terms of t_r), and express everything in terms of t_r:

$$t^2 - tt_r\left[2 + \left(\frac{\alpha d}{c}\right)^2\right] + \left[t_r^2 - \left(\frac{d}{c}\right)^2 - \frac{\alpha^2}{4}\left(\frac{d}{c}\right)^4\right] = 0 \Longrightarrow$$

$$t = \frac{1}{2}\left\{t_r\left[2 + \left(\frac{\alpha d}{c}\right)^2\right] \pm \sqrt{t_r^2\left[\cancel{4} + 4\left(\frac{\alpha d}{c}\right)^2 + \left(\frac{\alpha d}{c}\right)^4\right] - \cancel{4t_r^2} + 4\left(\frac{d}{c}\right)^2 + \alpha^2\left(\frac{d}{c}\right)^4}\right\}$$

$$= t_r\left[1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right] \pm \sqrt{\left[1 + (\alpha t_r)^2\right]\left(\frac{d}{c}\right)^2\left[1 + \left(\frac{\alpha d}{2c}\right)^2\right]}$$

Which sign? For small α we want $t \approx t_r + d/c$, so we need the + sign:

$$t = t_r\left[1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right] + \frac{d}{c}TD, \text{ where } D \equiv \sqrt{1 + \left(\frac{\alpha d}{2c}\right)^2}.$$

So $\mathscr{r} = c(t - t_r) \Rightarrow \mathscr{r} = \frac{ct_r}{2}\left(\frac{\alpha d}{c}\right)^2 + dTD$. Now go back to Eq. (★) and solve for $\sqrt{1 + (\alpha t)^2}$:

$$\begin{aligned}\sqrt{1 + (\alpha t)^2} &= \frac{1}{T}\left\{1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2 + \alpha^2 t_r\left[t_r\left(1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right) + \frac{d}{c}TD\right]\right\} \\ &= \frac{1}{T}\left\{\underbrace{\left[1 + (\alpha t_r)^2\right]}_{T^2}\left[1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right] + \frac{\alpha^2 t_r d}{c}TD\right\} = \left[1 + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right]T + \frac{\alpha^2 t_r d}{c}D.\end{aligned}$$

$$l = \frac{c}{\alpha}\left[\sqrt{1 + (\alpha t^2)} - \sqrt{1 + (\alpha t_r)^2}\right] = \frac{c}{\alpha}\left\{\left[\cancel{1} + \frac{1}{2}\left(\frac{\alpha d}{c}\right)^2\right]T + \frac{\alpha^2 t_r d}{c}D - \cancel{T}\right\} = \alpha d\left(\frac{d}{2c}T + t_r D\right).$$

Putting all this in, the numerator in square brackets in $\mathbf{F}$ becomes:

$$\begin{aligned}[\quad] &= \left\{c\alpha d\left(\frac{d}{2c}T + t_r D\right) - \frac{c\alpha t_r}{T}\left[\frac{ct_r}{2}\left(\frac{\alpha d}{c}\right)^2 + dTD\right]\right\}\left(c^2 - \frac{c^2\alpha^2 t_r^2}{T^2}\right) - c\frac{c\alpha}{T^3}d^2 \\ &= c\alpha d\left[\frac{d}{2c}T + \cancel{t_r D} - \frac{d(at_r)^2}{2cT} - \cancel{t_r D}\right]\frac{c^2}{T^2}\left[1 + (\cancel{\alpha t_r})^2 - (\cancel{\alpha t_r})^2\right] - \frac{c^2\alpha d^2}{T^3} \\ &= \frac{c^2\alpha d^2}{T^3}\left[\frac{1}{2}T^2 - \frac{1}{2}(\alpha t_r)^2 - 1\right] = \frac{c^2\alpha d^2}{2T^3}\left[1 + (\alpha t_r)^2 - (\alpha t_r)^2 - 2\right] = -\frac{c^2\alpha d^2}{2T^3}.\end{aligned}$$

$\therefore \mathbf{F} = \dfrac{q^2}{4\pi\epsilon_0}\dfrac{c^2\alpha d^2}{\left[(c\mathscr{r} - lv)T\right]^3}\hat{\mathbf{x}}.$ It remains to compute the denominator:

$$\begin{aligned}(c\mathscr{r} - lv)T &= \left\{c\left[\frac{ct_r}{2}\left(\frac{\alpha d}{c}\right)^2 + dTD\right] - \alpha d\left(\frac{d}{2c}T + t_r D\right)\frac{c\alpha t_r}{T}\right\}T \\ &= \left[\frac{1}{2}\alpha^2\cancel{t_r}d^2 + cdTD - \frac{1}{2}\alpha^2\cancel{t_r}d^2 - \frac{cd(\alpha t_r)^2}{T}D\right]T = cdD\Big[\underbrace{T^2 - (\alpha t_r)^2}_{1+(\cancel{\alpha t_r})^2-(\cancel{\alpha t_r})^2}\Big] = dcD.\end{aligned}$$

$$\therefore \mathbf{F} = \frac{q^2}{4\pi\epsilon_0}\frac{c^2 d^2\alpha}{c^3 d^3 D^3}\hat{\mathbf{x}} = \boxed{\frac{q^2}{4\pi\epsilon_0}\frac{\alpha}{cd\left[1 + (\alpha d/2c)^2\right]^{3/2}}\hat{\mathbf{x}}} \quad \left(\alpha = \frac{F}{mc}\right).$$

Energy must come from the "reservoir" of energy stored in the electromagnetic fields.

(b) $F = mc\alpha = \underset{\substack{\uparrow \\ \text{(force on one end only)}}}{\frac{1}{2}}\dfrac{q^2}{4\pi\epsilon_0}\dfrac{\alpha}{cd\left[1 + (\alpha d/2c)^2\right]^{3/2}} \Rightarrow \left[1 + \left(\dfrac{\alpha d}{2c}\right)^2\right]^{3/2} = \dfrac{q^2}{8\pi\epsilon_0 mc^2 d} = \dfrac{\mu_0 q^2}{8\pi m d}.$

$$\therefore \alpha = \frac{2c}{d}\sqrt{\left(\frac{\mu_0 q^2}{8\pi m d}\right)^{2/3} - 1}, \text{ so } \boxed{F = \frac{2mc^2}{d}\sqrt{\left(\frac{\mu_0 q^2}{8\pi m d}\right)^{2/3} - 1}.}$$

Problem 12.62

(a) $A^\mu = (V/c, A_x, A_y, A_z)$ is a 4-vector (like $x^\mu = (ct, x, y, z)$), so (using Eq. 12.19): $V = \gamma(\bar{V} + v\bar{A}_x)$. But $\bar{V} = 0$, and

$$\bar{A}_x = \frac{\mu_0}{4\pi}\frac{(\mathbf{m} \times \bar{\mathbf{r}})_x}{\bar{r}^3}.$$

Now $(\mathbf{m} \times \bar{\mathbf{r}})_x = m_y\bar{z} - m_z\bar{y} = m_y z - m_z y$. So

$$V = \gamma v \frac{\mu_0}{4\pi}\frac{(m_y z - m_z y)}{\bar{r}^3}.$$

Now $\bar{x} = \gamma(x - vt) = \gamma R_x$, $\bar{y} = y = R_y$, $\bar{z} = z = R_z$, where $\mathbf{R}$ is the vector (in $\mathcal{S}$) from the (instantaneous) location of the dipole to the point of observation. Thus

$$\bar{r}^2 = \gamma^2 R_x^2 + R_y^2 + R_z^2 = \gamma^2(R_x^2 + R_y^2 + R_z^2) + (1 - \gamma^2)(R_y^2 + R_z^2) = \gamma^2\left(R^2 - \frac{v^2}{c^2}R^2\sin^2\theta\right)$$

(where θ is the angle between $\mathbf{R}$ and the x axis, so that $R_y^2 + R_z^2 = R^2\sin^2\theta$).

$$\therefore V = \frac{\mu_0}{4\pi}\frac{v\gamma(m_y R_z - m_z R_y)}{\gamma^3 R^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}; \quad \text{but } \mathbf{v}\cdot(\mathbf{m} \times \mathbf{R}) = v(\mathbf{m} \times \mathbf{R})_x = v(m_y R_z - m_z R_y), \quad \text{so}$$

$$\boxed{V = \frac{\mu_0}{4\pi}\frac{\mathbf{v}\cdot(\mathbf{m} \times \mathbf{R})\left(1 - \frac{v^2}{c^2}\right)}{R^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}},}$$

or, using $\mu_0 = \frac{1}{\epsilon_0 c^2}$ and $\mathbf{v}\cdot(\mathbf{m} \times \mathbf{R}) = \mathbf{R}\cdot(\mathbf{v} \times \mathbf{m})$: $V = \dfrac{1}{4\pi\epsilon_0}\dfrac{\hat{\mathbf{R}}\cdot(\mathbf{v} \times \mathbf{m})\left(1 - \frac{v^2}{c^2}\right)}{c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$.

(b) In the nonrelativistic limit ($v^2 \ll c^2$):

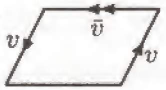
$$V = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\cdot(\mathbf{v} \times \mathbf{m})}{c^2R^2} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\cdot\mathbf{p}}{R^2}, \quad \text{with } \mathbf{p} = \frac{\mathbf{v} \times \mathbf{m}}{c^2},$$

which is the potential of an *electric* dipole.

Problem 12.63

(a) $\mathbf{B} = -\frac{\mu_0}{2}K\hat{\mathbf{y}}$ (Eq. 5.56); $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ (Eq. 6.1), so $\mathbf{N} = -\frac{\mu_0}{2}mK(\hat{\mathbf{z}} \times \hat{\mathbf{y}})$.

$\boxed{\mathbf{N} = \frac{\mu_0}{2}mK\hat{\mathbf{x}}} = \frac{\mu_0}{2}(\lambda v l^2)(\sigma v)\hat{\mathbf{x}} = \frac{\mu_0}{2}\lambda\sigma v^2 l^2\hat{\mathbf{x}}$.

(b) 

Charge density on the front side: λ_0 ($\lambda = \gamma\lambda_0$);
Charge density on the back side: $\bar{\lambda} = \bar{\gamma}\lambda_0$, where $\bar{v} = \frac{2v}{1+v^2/c^2} \Rightarrow$

$$\bar{\gamma} = \frac{1}{\sqrt{1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}}} = \frac{(1 + v^2/c^2)}{\sqrt{1 + 2\frac{v^2}{c^2} + \frac{v^4}{c^4} - 4\frac{v^2}{c^2}}} = \frac{1 + v^2/c^2}{\sqrt{1 - 2\frac{v^2}{c^2} + \frac{v^4}{c^4}}} = \frac{(1 + v^2/c^2)}{(1 - v^2/c^2)} = \gamma^2\left(1 + \frac{v^2}{c^2}\right).$$

Length of front and back sides in this frame: l/γ. So the net charge on the back side is:

$$q_+ = \bar{\lambda}\frac{l}{\gamma} = \gamma^2\left(1 + \frac{v^2}{c^2}\right)\frac{\lambda}{\gamma}\frac{l}{\gamma} = \left(1 + \frac{v^2}{c^2}\right)\lambda l.$$

Net charge on front side is:

$$q_- = \lambda_0 \frac{l}{\gamma} = \frac{\lambda}{\gamma}\frac{l}{\gamma} = \frac{1}{\gamma^2}\lambda l.$$

So the dipole moment (note: charges on *sides* are equal):

$$\mathbf{p} = (q_+)\frac{l}{2}\,\hat{\mathbf{y}} - (q_-)\frac{l}{2}\,\hat{\mathbf{y}} = \left[\left(1+\frac{v^2}{c^2}\right)\lambda l\frac{l}{2} - \frac{1}{\gamma^2}\lambda l\frac{l}{2}\right]\hat{\mathbf{y}} = \frac{\lambda l^2}{2}\left(1+\frac{v^2}{c^2}-1+\frac{v^2}{c^2}\right)\hat{\mathbf{y}} = \boxed{\frac{\lambda l^2 v^2}{c^2}\hat{\mathbf{y}}}.$$

$\mathbf{E} = \frac{\sigma_0}{2\epsilon_0}\hat{\mathbf{z}}$, where $\sigma = \gamma\sigma_0$, so $\mathbf{N} = \mathbf{p}\times\mathbf{E} = \frac{\lambda l^2 v^2}{c^2}\frac{\sigma}{2\epsilon_0\gamma}(\hat{\mathbf{y}}\times\hat{\mathbf{z}}) = \boxed{\frac{1}{\gamma}\frac{\mu_0}{2}\lambda\sigma l^2 v^2\hat{\mathbf{x}}.}$

So apart from the relativistic factor of γ the torque is the same in both systems—but in $\mathcal{S}$ it is the torque exerted by a *magnetic* field on a *magnetic* dipole, whereas in $\bar{\mathcal{S}}$ it is the torque exerted by an *electric* field on an *electric* dipole.

Problem 12.64

Choose axes so that **E** points in the z direction and **B** in the yz plane: $\mathbf{E} = (0,0,E)$; $\mathbf{B} = (0, B\cos\phi, B\sin\phi)$. Go to a frame moving at speed v in the x direction:

$$\bar{\mathbf{E}} = \big(0, -\gamma vB\sin\phi, \gamma(E+vB\cos\phi)\big);\quad \bar{\mathbf{B}} = \big(0, \gamma(B\cos\phi + \frac{v}{c^2}E), \gamma B\sin\phi\big).$$

(I used Eq. 12.108.) Parallel provided $\dfrac{-\gamma vB\sin\phi}{\gamma(B\cos\phi+\frac{v}{c^2}E)} = \dfrac{\gamma(E+vB\cos\phi)}{\gamma B\sin\phi}$, or

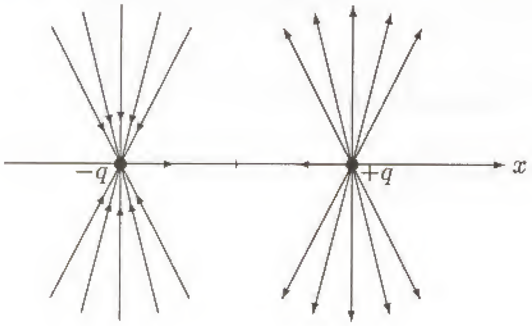
$$-vB^2\sin^2\phi = \big(B\cos\phi + \frac{v}{c^2}E\big)(E+vB\cos\phi) = EB\cos\phi + vB^2\cos^2\phi + \frac{v}{c^2}E^2 + \frac{v^2}{c^2}EB\cos\phi,$$

$$0 = vB^2 + \frac{v}{c^2}E^2 + EB\cos\phi\left(1+\frac{v^2}{c^2}\right);\qquad \frac{v}{1+v^2/c^2} = -\frac{EB\cos\phi}{B^2+E^2/c^2}.$$

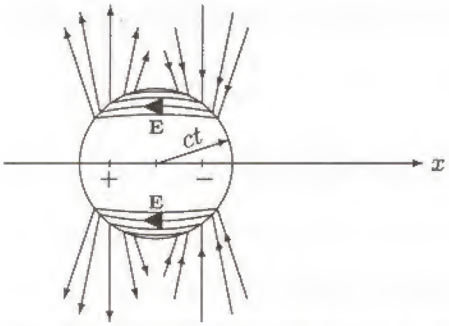
Now $\mathbf{E}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & E \\ 0 & B\cos\phi & B\sin\phi \end{vmatrix} = -EB\cos\phi\,\hat{\mathbf{x}}.$ So $\dfrac{\mathbf{v}}{1+v^2/c^2} = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2+E^2/c^2}.$ qed

$\boxed{\text{No,}}$ there can be no frame in which $\mathbf{E}\perp\mathbf{B}$, for $(\mathbf{E}\cdot\mathbf{B})$ is invariant, and since it is not zero in $\mathcal{S}$ it can't be zero in $\bar{\mathcal{S}}$.

Problem 12.65



Just before:
Field lines emanate
from *present* position
of particle.



Just after: Field lines *outside* sphere of radius ct emanate from position particle *would* have reached, had it kept going on its original "flight plan". *Inside* the sphere $E = 0$. On the surface the lines connect up (since they cannot simply *terminate* in empty space), as suggested in the figure.

This produces a dense cluster of tangentially-directed field lines, which expand with the spherical shell. This is a pictorial way of understanding the generation of *electromagnetic radiation.*

Problem 12.66

Equation 12.68 assumes the particle is (instantaneously) at rest in $\mathcal{S}$. Here the particle is at rest in $\bar{\mathcal{S}}$. So $\mathbf{F}_\perp = \frac{1}{\gamma}\bar{\mathbf{F}}_\perp$, $F_\parallel = \bar{F}_\parallel$. Using $\bar{\mathbf{F}} = q\bar{\mathbf{E}}$, then,

$$F_x = \bar{F}_x = q\bar{E}_x, \quad F_y = \frac{1}{\gamma}\bar{F}_y = \frac{1}{\gamma}q\bar{E}_y, \quad F_z = \frac{1}{\gamma}\bar{F}_z = \frac{1}{\gamma}q\bar{E}_z.$$

Invoking Eq. 12.108:

$$F_x = qE_x, \quad F_y = \frac{1}{\gamma}q\gamma(E_y - vB_z) = q(E_y - vB_z), \quad F_z = \frac{1}{\gamma}q\gamma(E_z + vB_y) = q(E_z + vB_y).$$

But $\mathbf{v}\times\mathbf{B} = -vB_z\,\hat{\mathbf{x}} + vB_y\,\hat{\mathbf{z}}$, so $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. qed

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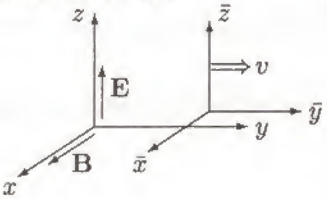
Problem 12.67



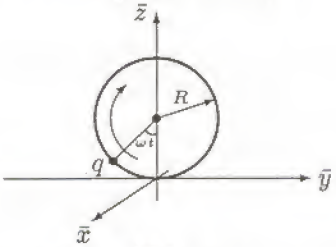
Rewrite Eq. 12.108 with $x \to y$, $y \to z$, $z \to x$:

$$\boxed{\begin{array}{lll} \bar{E}_y = E_y & \bar{E}_z = \gamma(E_z - vB_x) & \bar{E}_x = \gamma(E_x + vB_z) \\ \bar{B}_y = B_y & \bar{B}_z = \gamma\Big(B_z + \dfrac{v}{c^2}E_x\Big) & \bar{B}_x = \gamma\Big(B_x - \dfrac{v}{c^2}E_z\Big) \end{array}}$$

This gives the fields in system $\bar{\mathcal{S}}$ moving in the y direction at speed v.

Now $\mathbf{E} = (0, 0, E_0)$; $\mathbf{B} = (B_0, 0, 0)$, so $\bar{E}_y = 0$, $\bar{E}_z = \gamma(E_0 - vB_0)$, $\bar{E}_x = 0$.
If we want $\bar{\mathbf{E}} = 0$, we must pick v so that $E_0 - vB_0 = 0$; i.e. $\boxed{v = E_0/B_0.}$
(The condition $E_0/B_0 < c$ guarantees that there is no problem *getting* to such a system.)
With this, $\bar{B}_y = 0$, $\bar{B}_z = 0$, $\bar{B}_x = \gamma(B_0 - \frac{v}{c^2}E_0) = \gamma B_0(1 - \frac{v^2}{c^2}) = \gamma B_0\frac{1}{\gamma^2} = \frac{1}{\gamma}B_0$; $\boxed{\bar{\mathbf{B}} = \frac{1}{\gamma}B_0\,\hat{\mathbf{x}}.}$

The trajectory in $\bar{\mathcal{S}}$: Since the particle started out at rest at the origin in $\mathcal{S}$, it started out with velocity $-v\hat{\mathbf{y}}$ in $\bar{\mathcal{S}}$. According to Eq. 12.72 it will move in a circle of radius R, given by



$$p = qBR, \text{ or } \gamma m v = q\Big(\frac{1}{\gamma}B_0\Big)R \Rightarrow \boxed{R = \frac{m\gamma^2 v}{qB_0}.}$$

The actual trajectory is given by $\boxed{\bar{x} = 0\,;\ \bar{y} = -R\sin\omega\bar{t}\,;\ \bar{z} = R(1 - \cos\omega\bar{t});}$ where $\boxed{\omega = \frac{v}{R}.}$

The trajectory in $\mathcal{S}$: The Lorentz transformations Eqs. 12.18 and 12.19, for the case of relative motion in the y direction, read:

$$\begin{array}{ll} \bar{x} = x & x = \bar{x} \\ \bar{y} = \gamma(y - vt) & y = \gamma(\bar{y} + v\bar{t}) \\ \bar{z} = z & z = \bar{z} \\ \bar{t} = \gamma\left(t - \frac{v}{c^2}y\right) & t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{y}\right) \end{array}$$

So the trajectory in $\mathcal{S}$ is given by:

$$x = 0; \quad y = \gamma(-R\sin\omega\bar{t} + v\bar{t}) = \gamma\left\{-R\sin\left[\omega\gamma\left(t - \frac{v}{c^2}y\right)\right] + v\gamma\left(t - \frac{v}{c^2}y\right)\right\}, \text{ or}$$

$$\left.\underbrace{y\left(1 + \gamma^2\frac{v^2}{c^2}\right)}_{\gamma^2 y(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}) = \gamma^2 y} = \gamma^2 vt - \gamma R\sin\left[\omega\gamma\left(t - \frac{v}{c^2}y\right)\right]\right\} (y - vt)\gamma = -R\sin\left[\omega\gamma\left(t - \frac{v}{c^2}y\right)\right];$$

$$z = R(1 - \cos^2\omega\bar{t}) = R\left[1 - \cos\omega\gamma\left(t - \frac{v}{c^2}y\right)\right].$$

So: $\boxed{x = 0;\ y = vt - \dfrac{R}{\gamma}\sin\left[\omega\gamma\left(t - \dfrac{v}{c^2}y\right)\right];\ z = R - R\cos\left[\omega\gamma\left(t - \dfrac{v}{c^2}y\right)\right].}$

We can get rid of the trigonometric terms by the usual trick:

$$\left.\begin{array}{l} \gamma(y - vt) = -R\sin\left[\omega\gamma(t - \frac{v}{c^2}y)\right] \\ z - R = -R\cos\left[\omega\gamma(t - \frac{v}{c^2}y)\right] \end{array}\right\} \Rightarrow \boxed{\gamma^2(y - vt)^2 + (z - R)^2 = R^2.}$$

Absent the γ^2, this would be the cycloid we found back in Ch. 5 (Eq. 5.9). The γ^2 makes it, as it were, an *elliptical* cycloid — same picture as p. 206, but with the horizontal axis stretched out.

Problem 12.68

(a) $\left.\begin{array}{l} \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \text{ suggests } \mathbf{E} \to \frac{1}{\epsilon_0}\mathbf{D} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \text{ suggests } \mathbf{B} \to \mu_0\mathbf{H} \end{array}\right\}$ but it's a little cleaner if we divide by μ_0 while we're at it, so that $\mathbf{E} \to \frac{1}{\mu_0\epsilon_0}\mathbf{D} = c^2\mathbf{D}$, $\mathbf{B} \to \mathbf{H}$. Then: $\boxed{D^{\mu\nu} = \left\{\begin{array}{cccc} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{array}\right\}}$

Then (following the derivation on p. 539):

$$\frac{\partial}{\partial x^\nu}D^{0\nu} = c\,\boldsymbol{\nabla}\cdot\mathbf{D} = c\rho_f = J_f^0\ ;\ \frac{\partial}{\partial x^\nu}D^{1\nu} = \frac{1}{c}\frac{\partial}{\partial t}(-cD_x) + (\boldsymbol{\nabla}\times\mathbf{H})_x = (J_f)_x\ ;\ \text{so } \boxed{\frac{\partial D^{\mu\nu}}{\partial x^\nu} = J_f^\mu,}$$

where $\boxed{J_f^\mu = (c\rho_f, \mathbf{J}_f).}$ Meanwhile, the homogeneous Maxwell equations ($\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, $\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$) are unchanged, and hence $\boxed{\dfrac{\partial G^{\mu\nu}}{\partial x^\nu} = 0.}$

(b) $\boxed{H^{\mu\nu} = \left\{\begin{array}{cccc} 0 & H_x & H_y & H_z \\ -H_x & 0 & -cD_z & cD_y \\ -H_y & cD_z & 0 & -cD_x \\ -H_z & -cD_y & cD_x & 0 \end{array}\right\}}$

(c) If the material is at rest, $\eta_\nu = (-c, 0, 0, 0)$, and the sum over ν collapses to a single term:

$$D^{\mu 0}\eta_0 = c^2\epsilon F^{\mu 0}\eta_0 \Rightarrow D^{\mu 0} = c^2\epsilon F^{\mu 0} \Rightarrow -c\mathbf{D} = -c^2\epsilon\frac{\mathbf{E}}{c} \Rightarrow \mathbf{D} = \epsilon\mathbf{E} \text{ (Eq. 4.32)}, \checkmark$$

$$H^{\mu 0}\eta_0 = \frac{1}{\mu}G^{\mu 0}\eta_0 \Rightarrow H^{\mu 0} = \frac{1}{\mu}G^{\mu 0} \Rightarrow -\mathbf{H} = -\frac{1}{\mu}\mathbf{B} \Rightarrow \mathbf{H} = \frac{1}{\mu}\mathbf{B} \text{ (Eq. 6.31)}. \checkmark$$

(d) In general, $\eta_\nu = \gamma(-c, \mathbf{u})$, so, for $\mu = 0$:

$$D^{0\nu}\eta_\nu = D^{01}\eta_1 + D^{02}\eta_2 + D^{03}\eta_3 = cD_x(\gamma u_x) + cD_y(\gamma u_y) + cD_z(\gamma u_z) = \gamma c(\mathbf{D}\cdot\mathbf{u}),$$

$$F^{0\nu}\eta_\nu = F^{01}\eta_1 + F^{02}\eta_2 + F^{03}\eta_3 = \frac{E_x}{c}(\gamma u_x) + \frac{E_y}{c}(\gamma u_y) + \frac{E_z}{c}(\gamma u_z) = \frac{\gamma}{c}(\mathbf{E}\cdot\mathbf{u}), \text{ so}$$

$$D^{0\nu}\eta_\nu = c^2\epsilon F^{0\nu}\eta_\nu \Rightarrow \gamma c(\mathbf{D}\cdot\mathbf{u}) = c^2\epsilon\left(\frac{\gamma}{c}\right)(\mathbf{E}\cdot\mathbf{u}) \Rightarrow \mathbf{D}\cdot\mathbf{u} = \epsilon(\mathbf{E}\cdot\mathbf{u}). \qquad [1]$$

$$H^{0\nu}\eta_\nu = H^{01}\eta_1 + H^{02}\eta_2 + H^{03}\eta_3 = H_x(\gamma u_x) + H_y(\gamma u_y) + H_z(\gamma u_z) = \gamma(\mathbf{H}\cdot\mathbf{u}),$$

$$G^{0\nu}\eta_\nu = G^{01}\eta_1 + G^{02}\eta_2 + G^{03}\eta_3 = B_x(\gamma u_x) + B_y(\gamma u_y) + B_z(\gamma u_z) = \gamma(\mathbf{B}\cdot\mathbf{u}), \text{ so}$$

$$H^{0\nu}\eta_\nu = \frac{1}{\mu}G^{0\nu}\eta_\nu \Rightarrow \gamma(\mathbf{H}\cdot\mathbf{u}) = \frac{1}{\mu}(\gamma)(\mathbf{B}\cdot\mathbf{u}) \Rightarrow \mathbf{H}\cdot\mathbf{u} = \frac{1}{\mu}(\mathbf{B}\cdot\mathbf{u}). \qquad [2]$$

Similarly, for $\mu = 1$:

$$\begin{aligned}
D^{1\nu}\eta_\nu &= D^{10}\eta_0 + D^{12}\eta_2 + D^{13}\eta_3 = (-cD_x)(-\gamma c) + H_z(\gamma u_y) + (-H_y)(\gamma u_z) = \gamma(c^2 D_x + u_y H_z - u_z H_y) \\
&= \gamma\left[c^2\mathbf{D} + (\mathbf{u}\times\mathbf{H})\right]_x, \\
F^{1\nu}\eta_\nu &= F^{10}\eta_0 + F^{12}\eta_2 + F^{13}\eta_3 = \frac{-E_x}{c}(-\gamma c) + B_z(\gamma u_y) + (-B_y)(\gamma u_z) = \gamma(E_x + u_y B_z - u_z B_y) \\
&= \gamma\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right]_x, \text{ so } D^{1\nu}\eta_\nu = c^2\epsilon F^{1\nu}\eta_\nu \Rightarrow
\end{aligned}$$

$$\gamma\left[c^2\mathbf{D} + (\mathbf{u}\times\mathbf{H})\right]_x = c^2\epsilon(\gamma)\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right]_x \Rightarrow \mathbf{D} + \frac{1}{c^2}(\mathbf{u}\times\mathbf{H}) = \epsilon\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right]. \qquad [3]$$

$$\begin{aligned}
H^{1\nu}\eta_\nu &= H^{10}\eta_0 + H^{12}\eta_2 + H^{13}\eta_3 = (-H_x)(-\gamma c) + (-cD_z)(\gamma u_y) + (cD_y)(\gamma u_z) \\
&= \gamma c(H_x - u_y D_z + u_z D_y) = \gamma c\left[\mathbf{H} - (\mathbf{u}\times\mathbf{D})\right]_x, \\
G^{1\nu}\eta_\nu &= G^{10}\eta_0 + G^{12}\eta_2 + G^{13}\eta_3 = (-B_x)(-\gamma c) + \left(-\frac{E_z}{c}\right)(\gamma u_y) + \left(\frac{E_y}{c}\right)(\gamma u_z) \\
&= \frac{\gamma}{c}(c^2 B_x - u_y E_z + u_z E_y) = \frac{\gamma}{c}\left[c^2\mathbf{B} - (\mathbf{u}\times\mathbf{E})\right]_x, \text{ so } H^{1\nu}\eta_\nu = \frac{1}{\mu}G^{1\nu}\eta_\nu \Rightarrow
\end{aligned}$$

$$\gamma c\left[\mathbf{H} - (\mathbf{u}\times\mathbf{D})\right]_x = \frac{1}{\mu}\frac{\gamma}{c}\left[c^2\mathbf{B} - (\mathbf{u}\times\mathbf{E})\right]_x \Rightarrow \mathbf{H} - (\mathbf{u}\times\mathbf{D}) = \frac{1}{\mu}\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right]. \qquad [4]$$

Use Eq. [4] as an expression for $\mathbf{H}$, plug this into Eq. [3], and solve for $\mathbf{D}$:

$$\mathbf{D} + \frac{1}{c^2}\mathbf{u}\times\left\{(\mathbf{u}\times\mathbf{D}) + \frac{1}{\mu}\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right]\right\} = \epsilon\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right];$$

$$\mathbf{D} + \frac{1}{c^2}\left[(\mathbf{u}\cdot\mathbf{D})\mathbf{u} - u^2\mathbf{D}\right] = \epsilon\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right] - \frac{1}{\mu c^2}(\mathbf{u}\times\mathbf{B}) + \frac{1}{\mu c^4}\left[\mathbf{u}\times(\mathbf{u}\times\mathbf{E})\right].$$

Using Eq. [1] to rewrite $\mathbf{u}\cdot\mathbf{D}$:

$$\begin{aligned}\mathbf{D}\left(1-\frac{u^2}{c^2}\right) &= -\frac{\epsilon}{c^2}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}+\epsilon\left[\mathbf{E}+(\mathbf{u}\times\mathbf{B})\right]-\frac{1}{\mu c^2}(\mathbf{u}\times\mathbf{B})+\frac{1}{\mu c^4}\left[(\mathbf{E}\cdot\mathbf{u})\mathbf{u}-u^2\mathbf{E}\right]\\ &= \epsilon\left\{\left[1-\frac{u^2}{\epsilon\mu c^4}\right]\mathbf{E}-\frac{1}{c^2}\left[1-\frac{1}{\epsilon\mu c^2}\right](\mathbf{E}\cdot\mathbf{u})\mathbf{u}+(\mathbf{u}\times\mathbf{B})\left[1-\frac{1}{\epsilon\mu c^2}\right]\right\}.\end{aligned}$$

Let $\boxed{\gamma\equiv\frac{1}{\sqrt{1-u^2/c^2}},\quad v\equiv\frac{1}{\sqrt{\epsilon\mu}}.}$ Then

$$\boxed{\mathbf{D}=\gamma^2\epsilon\left\{\left(1-\frac{u^2v^2}{c^4}\right)\mathbf{E}+\left(1-\frac{v^2}{c^2}\right)\left[(\mathbf{u}\times\mathbf{B})-\frac{1}{c^2}(\mathbf{E}\cdot\mathbf{u})\mathbf{u}\right]\right\}.}$$

Now use Eq. [3] as an expression for $\mathbf{D}$, plug this into Eq. [4], and solve for $\mathbf{H}$:

$$\mathbf{H}-\mathbf{u}\times\left\{-\frac{1}{c^2}(\mathbf{u}\times\mathbf{H})+\epsilon\left[\mathbf{E}+(\mathbf{u}\times\mathbf{B})\right]\right\}=\frac{1}{\mu}\left[\mathbf{B}-\frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right];$$

$$\mathbf{H}+\frac{1}{c^2}\left[(\mathbf{u}\cdot\mathbf{H})\mathbf{u}-u^2\mathbf{H}\right]=\frac{1}{\mu}\left[\mathbf{B}-\frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right]+\epsilon(\mathbf{u}\times\mathbf{E})+\epsilon\left[\mathbf{u}\times(\mathbf{u}\times\mathbf{B})\right].$$

Using Eq. [2] to rewrite $\mathbf{u}\cdot\mathbf{H}$:

$$\begin{aligned}\mathbf{H}\left(1-\frac{u^2}{c^2}\right) &= -\frac{1}{\mu c^2}(\mathbf{B}\cdot\mathbf{u})\mathbf{u}+\frac{1}{\mu}\left[\mathbf{B}-\frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right]+\epsilon(\mathbf{u}\times\mathbf{E})+\epsilon\left[(\mathbf{B}\cdot\mathbf{u})\mathbf{u}-u^2\mathbf{B}\right]\\ &= \frac{1}{\mu}\left\{\left[1-\mu\epsilon u^2\right]\mathbf{B}+\left(\epsilon\mu-\frac{1}{c^2}\right)\left[(\mathbf{u}\times\mathbf{E})+(\mathbf{B}\cdot\mathbf{u})\mathbf{u}\right]\right\}.\end{aligned}$$

$$\boxed{\mathbf{H}=\frac{\gamma^2}{\mu}\left\{\left(1-\frac{u^2}{v^2}\right)\mathbf{B}+\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\left[(\mathbf{u}\times\mathbf{E})+(\mathbf{B}\cdot\mathbf{u})\mathbf{u}\right]\right\}.}$$

Problem 12.69

We know that (proper) power transforms as the zeroth component of a 4-vector: $K^0=\frac{1}{c}\frac{dW}{d\tau}$. The Larmor formula says that for $v=0$, $\frac{dW}{dt}=\frac{\mu_0q^2a^2}{6\pi c}$ (Eq. 11.70). Can we think of a 4-vector whose zeroth component reduces to this when the velocity is zero?

Well, a^2 smells like $(\alpha^\nu\alpha_\nu)$, but how do we get a 4-vector in here? How about η^μ, whose zeroth component is just c, when $v=0$? Try, then:

$$K^\mu=\frac{\mu_0q^2}{6\pi c^3}(\alpha^\nu\alpha_\nu)\eta^\mu.$$

This has the right transformation properties, but we must check that it does reduce to the Larmor formula when $v\to0$:

$\frac{dW}{dt}=\frac{1}{\gamma}\frac{dW}{d\tau}=\frac{1}{\gamma}cK^0=\frac{1}{\gamma}c\frac{\mu_0q^2}{6\pi c^3}(\alpha^\nu\alpha_\nu)\eta^0$, but $\eta^0=c\gamma$, so $\boxed{\frac{dW}{dt}=\frac{\mu_0q^2}{6\pi c}(\alpha^\nu\alpha_\nu).}$ [Incidentally, this tells us that the power itself (as opposed to *proper* power) is a *scalar*. If this had been obvious from the start, we could simply have looked for a Lorentz *scalar* that generalizes the Larmor formula.]

In Prob. 12.38(b) we calculated $(\alpha^\nu \alpha_\nu)$ in terms of the *ordinary* velocity and acceleration:

$$\alpha^\nu \alpha_\nu = \gamma^4\left[a^2 + \frac{(\mathbf{v}\cdot\mathbf{a})^2}{(c^2 - v^2)}\right] = \gamma^6\left[a^2\gamma^{-2} + \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\right]$$

$$= \gamma^6\left[a^2\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\right] = \gamma^6\left\{a^2 - \frac{1}{c^2}\left[v^2a^2 - (\mathbf{v}\cdot\mathbf{a})^2\right]\right\}.$$

Now $\mathbf{v}\cdot\mathbf{a} = va\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{a}$, so:

$$v^2a^2 - (\mathbf{v}\cdot\mathbf{a})^2 = v^2a^2(1-\cos^2\theta) = v^2a^2\sin^2\theta = |\mathbf{v}\times\mathbf{a}|^2.$$

$$\alpha^\nu\alpha_\nu = \gamma^6\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right).$$

$\boxed{\dfrac{dW}{dt} = \dfrac{\mu_0 q^2}{6\pi c}\gamma^6\left(a^2 - \left|\dfrac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right),}$ which is Liénard's formula (Eq. 11.73).

Problem 12.70

(a) It's inconsistent with the constraint $\eta_\mu K^\mu = 0$ (Prob. 12.38(d)).

(b) We want to find a 4-vector b^μ with the property that $(\frac{d\alpha^\mu}{d\tau} + b^\mu)\eta_\mu = 0$. How about $b^\mu = \kappa(\frac{d\alpha^\nu}{d\tau}\eta_\nu)\eta^\mu$? Then $(\frac{d\alpha^\mu}{d\tau} + b^\mu)\eta_\mu = \frac{d\alpha^\mu}{d\tau}\eta_\mu + \kappa\frac{d\alpha^\nu}{d\tau}\eta_\nu(\eta^\mu\eta_\mu)$. But $\eta^\mu\eta_\mu = -c^2$, so this becomes $(\frac{d\alpha^\mu}{d\tau}\eta_\mu) - c^2\kappa(\frac{d\alpha^\nu}{d\tau}\eta_\nu)$, which *is* zero, if we pick $\kappa = 1/c^2$. This suggests $\boxed{K^\mu_{\text{rad}} = \dfrac{\mu_0 q^2}{6\pi c}\left(\dfrac{d\alpha^\mu}{d\tau} + \dfrac{1}{c^2}\dfrac{d\alpha^\nu}{d\tau}\eta_\nu\eta^\mu\right).}$ Note that $\eta^\mu = (c, \mathbf{v})\gamma$, so the spatial components of b^μ vanish in the nonrelativistic limit $v \ll c$, and hence this still reduces to the Abraham-Lorentz formula. [Incidentally, $\alpha^\nu\eta_\nu = 0 \Rightarrow \frac{d}{d\tau}(\alpha^\nu\eta_\nu) = 0 \Rightarrow \frac{d\alpha^\nu}{d\tau}\eta_\nu + \alpha^\nu\frac{d\eta_\nu}{d\tau} = 0$, so $\frac{d\alpha^\nu}{d\tau}\eta_\nu = -\alpha^\nu\alpha_\nu$, and hence b^μ can just as well be written $-\frac{1}{c^2}(\alpha^\nu\alpha_\nu)\eta^\mu$.]

Problem 12.71

Define the electric current 4-vector as before: $J^\mu_e = (c\rho_e, \mathbf{J}_e)$, and the magnetic current the same way: $J^\mu_m = (c\rho_m, \mathbf{J}_m)$. The fundamental laws are then

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu_e, \quad \partial_\nu G^{\mu\nu} = \frac{\mu_0}{c}J^\mu_m, \quad K^\mu = \left(q_e F^{\mu\nu} + \frac{q_m}{c}G^{\mu\nu}\right)\eta_\nu.}$$

The first of these reproduces $\nabla\cdot\mathbf{E} = (1/\epsilon_0)\rho_e$ and $\nabla\times\mathbf{B} = \mu_0\mathbf{J}_e + \mu_0\epsilon_0\partial\mathbf{E}/\partial t$, just as before (p. 539); the second yields $\nabla\cdot\mathbf{B} = (\mu_0/c)(c\rho_m) = \mu_0\rho_m$ and $-(1/c)(\partial\mathbf{B}/\partial t + \nabla\times\mathbf{E}) = (\mu_0/c)\mathbf{J}_m$, or $\nabla\times\mathbf{E} = -\mu_0\mathbf{J}_m - \partial\mathbf{B}/\partial t$ (generalizing page 540). These are Maxwell's equations with magnetic charge (Eq. 7.43). The third (following the argument on p. 540) says

$$\begin{aligned}
K^1 &= \frac{q_e}{\sqrt{1-u^2/c^2}}\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right]_x + \frac{q_m}{c}\left[\frac{-c}{\sqrt{1-u^2/c^2}}(-B_x) + \frac{u_y}{\sqrt{1-u^2/c^2}}\left(-\frac{E_z}{c}\right) + \frac{u_z}{\sqrt{1-u^2/c^2}}\left(\frac{E_y}{c}\right)\right], \\
\mathbf{K} &= \frac{1}{\sqrt{1-u^2/c^2}}\left\{q_e\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right] + q_m\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right]\right\}, \text{ or} \\
\mathbf{F} &= q_e\left[\mathbf{E} + (\mathbf{u}\times\mathbf{B})\right] + q_m\left[\mathbf{B} - \frac{1}{c^2}(\mathbf{u}\times\mathbf{E})\right],
\end{aligned}$$

which is the generalized Lorentz force law (Eq. 7.69).